Scheduling Multi-User Periodic Arrival of Tasks : Two Linear Programming Formulations

Emmanuel Medernach[1] Philippe Lacomme[1] Eric Sanlaville[2]
Claire Hanen[3]

[1]medernac@clermont.in2p3.fr, placomme@isima.fr
LPC - IN2P3 and LIMOS, Blaise Pascal University of Clermont-Ferrand

[2]Eric.Sanlaville@univ-lehavre.fr
LITIS, University of Le Havre

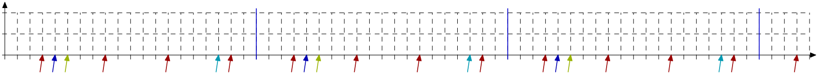
[3]Claire.Hanen@lip6.fr
LIP6, Université de Paris 10

ASTEC 2009

Outline

Cyclic scheduling of *periodically occurring* tasks

- A generic task is a task which must be performed infinitely often.
- Each occurrence of a generic task is periodically released.



Known result

If a schedule exists, then there exists a feasible schedule which is cyclic with a period T equal to the least common multiple of the periods of the individual tasks. ["Scheduling Periodically Occurring Tasks on Multiple Processors" - Lawler, Martel (1980)]

We will suppose that all tasks appear in the first period.

Cyclic scheduling of periodically occuring tasks

Notations

There are M identical machines.
A generic task i has

- a task period α_i
- a release time r_i
- and a required processing time p_i.

The *schedule period* is T (a multiple of all α_i).
The j^{th} occurrence of task i is denoted by $\langle i, j \rangle$ and has a release date $r_i + (j-1)\alpha_i$. The number of occurence of the task i in a period is T/α_i.

Condition of feasibility

$$\sum_i \frac{p_i}{\alpha_i} \leq M$$

This condition is necessary. It is also sufficient ! (Mc Naughton: preemption at the borders mean several occurrences of the task)

Periodic schedule

Among possible schedules we seek periodic schedules :

Definition (periodic schedule)

A schedule is periodic if all tasks run periodically (but not necessarily on the same machine each period). The steady state begins when all generic tasks have been executed at least once. Time before the steady state is called the transitory phase.

- Easy to implement.
- The problem is back to a finite problem
- But not necessarily dominant (depending on the chosen criterion)

Steady state

Steady state

The steady state is reached on a period if and only if this period contains the total amount of computation received during each period which is $\sum_i T \frac{p_i}{\alpha_i}$.

As soon as a period is complete in a periodic schedule it defined a pattern which must repeat itself in all following periods.

Definition (Pattern of a periodic schedule)

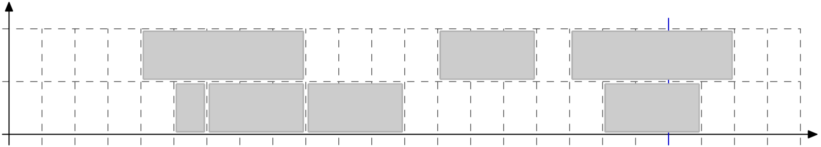
The pattern of a periodic schedule is equal to the beginning execution times of all jobs received during a whole period in the steady state.

Note: A periodic schedule defines a pattern but a pattern defines many schedules, because occurrence numbers of a task in a pattern may be shifted without changing the schedule pattern.

Pattern

Example of a pattern with 2 machines and a global period of 20

$$
\begin{array}{rclrclrcl}
r_1 & = & 3, & p_1 & = & 3 & t_1 & = & 6 \\
r_2 & = & 8, & p_2 & = & 3 & t_2 & = & 9 \\
r_3 & = & 13, & p_3 & = & 3 & t_3 & = & 13 \\
r_4 & = & 18, & p_4 & = & 3 & t_4 & = & 18 \\
r_5 & = & 4, & p_5 & = & 4 & t_5 & = & 4 \\
r_6 & = & 5, & p_6 & = & 1 & t_6 & = & 5 \\
r_7 & = & 17, & p_7 & = & 5 & t_7 & = & 17
\end{array}
$$



Multi-User Problem

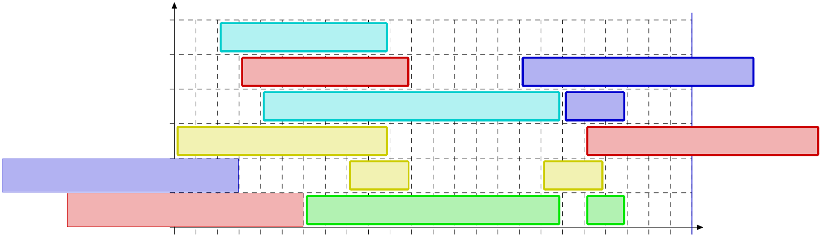
- There are many users sending tasks and sharing a computing facility.
- Each task belongs to one user.
- All users could use all available machines.
- OBJECTIVE: To give each user a fair share of the schedule.
- A comparison criterion is used to compare user share of the schedule.

Comparison criterion

We wish to enforce this following independance property on comparison criterion:

Independance of users in the comparison criterion

The comparision criterion used to evaluate a user share of the schedule must depends only on the schedule of this user jobs.



Criterion used: Presence time

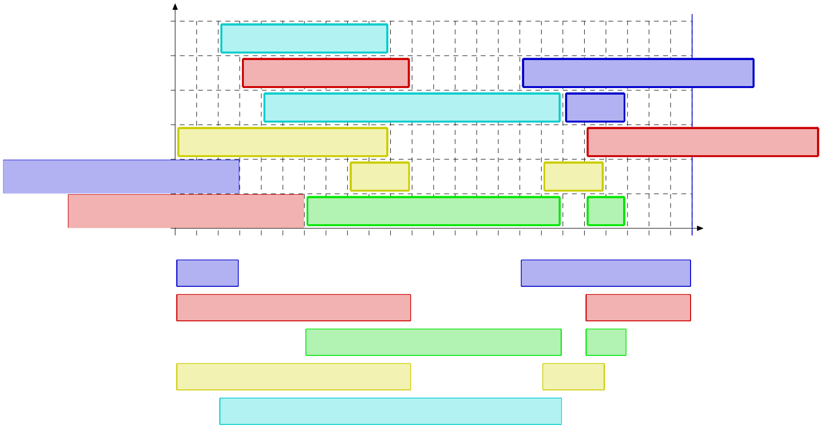
Definition (Presence Intervals)

A given user U is present at time t if there are tasks of U waiting or running at time t.
Presence time for a given user is the length of presence intervals for this user in a period after the steady state (because this is the dominant part).

The comparison criterion will depend on the presence times. It reduces to *Flow Time* if each user has only one task.

- **MeanPresenceTime** (or total presence time)
- **MinMaxPresenceTime**
- **WeightedMeanPresenceTime**: We add a weight equal to the total amount of computing of user job. (Stretch)
- **WeightedMinMaxPresenceTime**

Criterion used: Presence time (example)



Pattern: Transitory phase

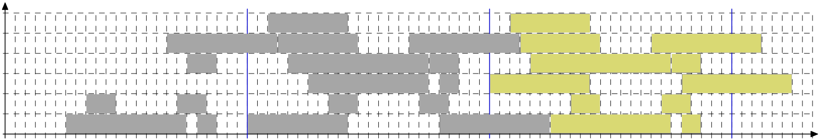
Dominance

For a given pattern there is only one dominant schedule (modulo tasks locations on machines) for presence interval based criterions.

Given an arbitrary pattern we are able to reach it :

Reachability of a given pattern

It is always possible to reach the steady state after at most $2 + M$ periods or after at most 3 periods if all jobs have length less than the global period.



Transitory phase model

- We wish to model a periodic schedule until it finally reaches the steady state.
- The number of time intervals of duration T is fixed to a and we impose that the steady state is reached at the last period $[(a-1)T, aT]$.
- If the problem becomes impossible we try again with $a+1$. We know that it will finally works with a at most $2+M$.
- We will use integer programming techniques.

Model formulation: Data

- We have M identical machines.
- The global schedule period is T, which is a multiple of all α_i. The number of tasks i released during a period T is $K_i = T/\alpha_i$.
- The generic task i belongs to the user U_i, has period α_i and a first release date r_i such that the release of task $\langle i,j \rangle$ is $r_{\langle i,j \rangle} = r_i + (j-1)\alpha_i$.
- We have a fixed number of periods a and we impose that the steady state is reached at the last period.

H is a given sufficiently high constant.

Model formulation: Variables

- The execution time of task $\langle i, j \rangle$ is denoted $t_{\langle i,j \rangle}$.
- $m_{i,j,k}$ is a binary variable which is equal to 1 if and only if task $\langle i, j \rangle$ runs on machine l, and else it is equal to 0.
- $x_{i_1,j_1,i2,j2}$ is a binary variables which is equal to 1 if and only if task $\langle i_1, j_1 \rangle$ runs before task $\langle i_2, j_2 \rangle$ (not necessarily on the same machine), and else it is equal to 0.

Constraints

For all tasks $\langle i,j \rangle$:

- We must respect release times: $t_{\langle i,j \rangle} \geq r_{\langle i,j \rangle}$ and order of tasks: $t_{\langle i,j \rangle} \leq t_{\langle i,j+1 \rangle}$
- All tasks must be allocated to one machine: $\sum_{l=1}^{M} m_{i,j,l} = 1$
- All tasks are periodic: $t_{\langle i,j+K_i \rangle} = t_{\langle i,j \rangle} + T$

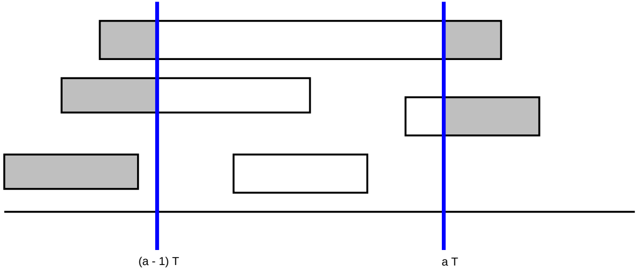
For all tasks $\langle i_1,j_1 \rangle \neq \langle i_2,j_2 \rangle$:

- Either a task begins before or after another one: $x_{i_2,j_2,i_1,j_1} + x_{i_1,j_1,i_2,j_2} = 1$
- $t_{\langle i_2,j_2 \rangle} \leq t_{\langle i_1,j_1 \rangle} + Hx_{i_1,j_1,i_2,j_2}$
- If tasks are allocated on the same machine, then it must not overlap: $\forall l \in \{1 \ldots M\}, t_{\langle i_1,j_1 \rangle} + p_{i_1} \leq t_{\langle i_2,j_2 \rangle} + H(3 - x_{i_1,j_1,i_2,j_2} - m_{i_1,j_1,l} - m_{i_2,j_2,l})$
- Order is the same between global periods: $x_{i_1,j_1,i_2,j_2} = x_{i_1,j_1+K_1,i_2,j_2+K_2}$

Linearization of Presence Time

We cut tasks to look only inside the last interval $[(a-1)T, aT]$.
For that we apply to all tasks the function

$$x \mapsto \max(\min(x, aT), (a-1)T)$$



This allows to compute the amount of tasks inside the last period, we enforce that this quantity is equal to $\sum_i T \frac{p_i}{\alpha_i}$.

Linearization of Presence Time

μ is a binary variable such that $\mu = 1$ if $y \leq x$ and else $\mu = 0$.

$$\begin{aligned} \mu H + y &\geq x \\ (1-\mu)H + x &\geq y \end{aligned}$$

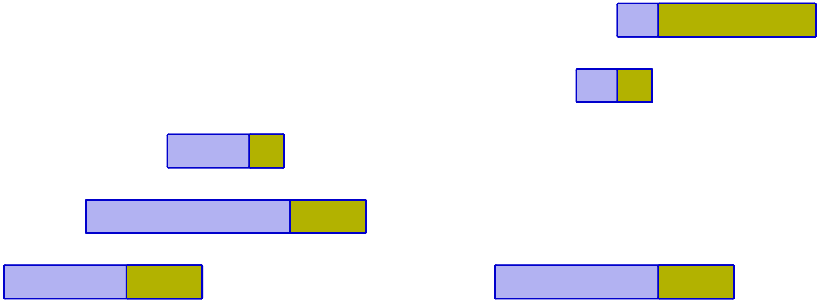
Minimum: $z = \min(x, y)$

$$\begin{aligned} z &\leq x \\ z &\leq y \\ (1-\mu)H + z &\geq y \\ \mu H + z &\geq x \end{aligned}$$

Maximum: $z = \max(x, y)$

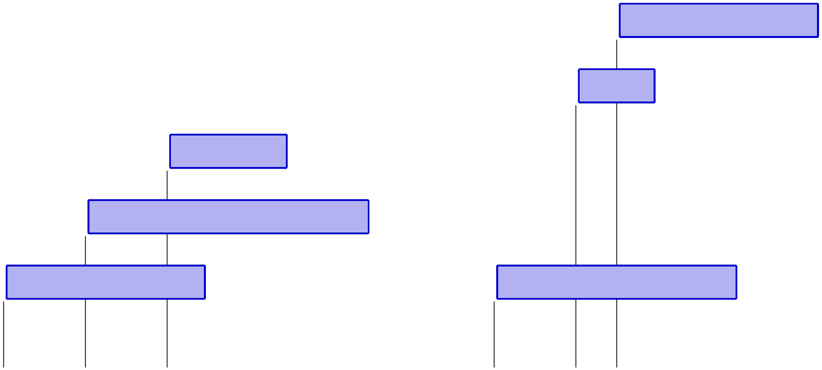
$$\begin{aligned} z &\geq x \\ z &\geq y \\ (1-\mu)H + x &\geq z \\ \mu H + y &\geq z \end{aligned}$$

Linearization of Presence Time



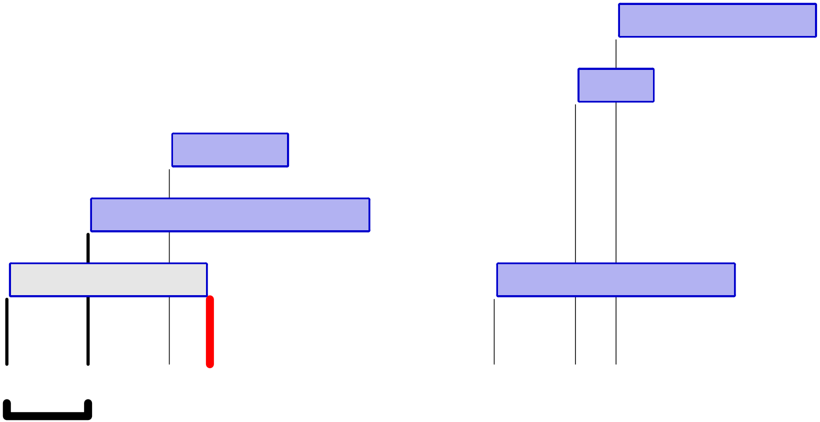
We extract for each task of a given user his tasks presence times.
Blue part is the waiting interval, Orange part is the execution interval.

Linearization of Presence Time



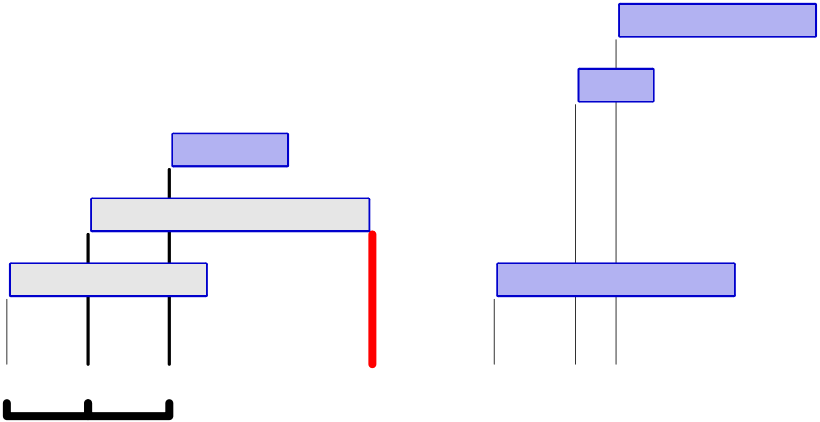
Release dates are sorted in non-decreasing order.

Linearization of Presence Time



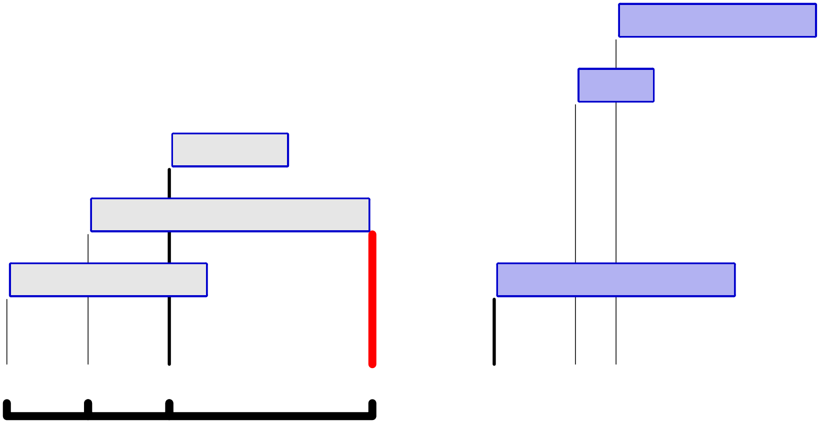
$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

Linearization of Presence Time



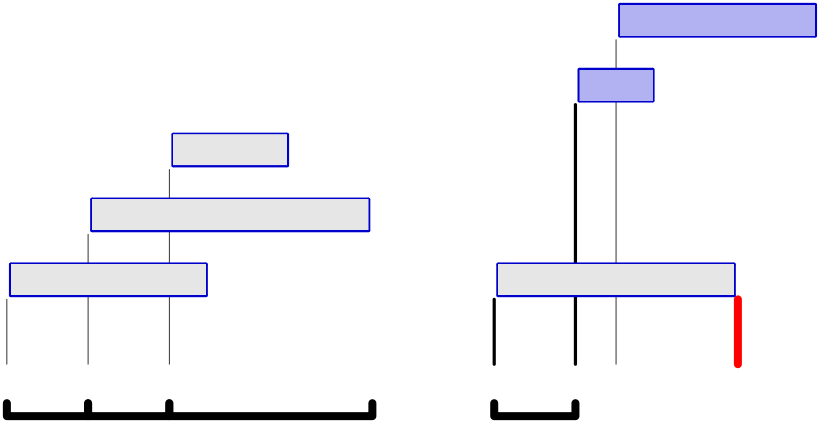
$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

Linearization of Presence Time



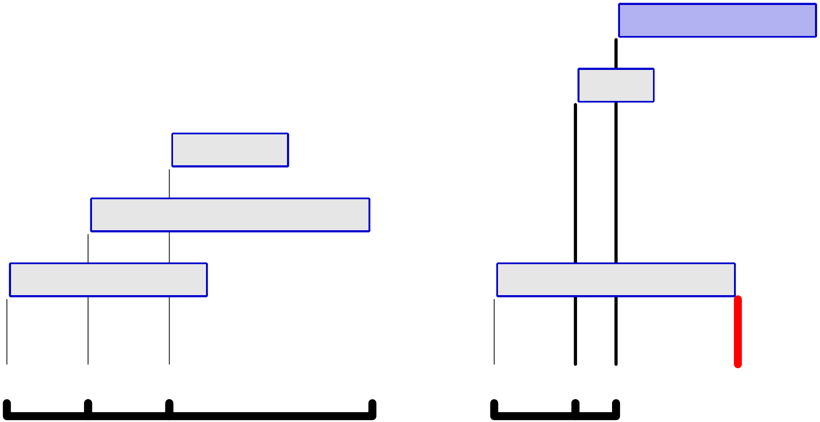
$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

Linearization of Presence Time



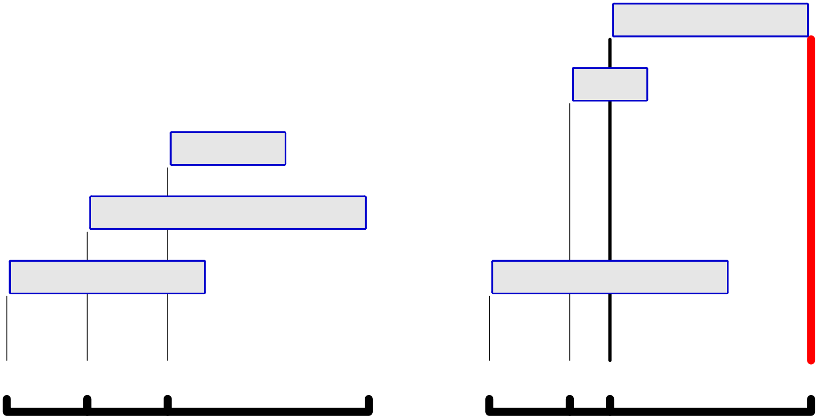
$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

Linearization of Presence Time



$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

Linearization of Presence Time



$$\sum_i \min(maxend_i, r_{i+1}) - r_i$$

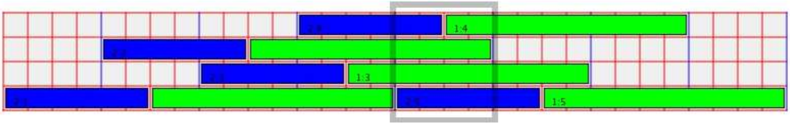
Results

2 users and 4 machines.
User U_1: $\alpha_1 = 4$, $p_1 = 10$, $r_1 = 4$
User U_2: $\alpha_2 = 4$, $p_2 = 6$, $r_2 = 0$

Periods	5
MinMaxPresence	27s
WeightedMinMaxPresence	1.7s
MeanPresence	24s
WeightedMeanPresence	16s

Table: Execution times

WeightedMeanPresence, $a = 5$:



Results

3 users and 3 machines
User U_1: $\alpha_1 = 3$, $p_1 = 2$, $r_1 = 0$
User U_2: $\alpha_2 = 6$, $p_2 = 4$, $r_2 = 0$
User U_3: $\alpha_3 = 6$, $p_3 = 8$, $r_3 = 0$

Periods	3	4
MinMaxPresence	1s	21s
WeightedMinMaxPresence	5m15	7m43
MeanPresence	2m10	72m4
WeightedMeanPresence	28s	4m20

Table: Execution times

MeanPresence, $a = 4$:

Pattern based model

- This second formulation works directly on patterns.
- We model pattern and evaluate its performances for all users.
- If a pattern is found we know how to reconstruct a schedule based on it. We could also use the pattern found inside the first model to compute the best possible transitory phase.
- Inside a period of the steady state, a task may be cut in pieces.
- In the steady state the machines must process all the computing quantity received during a period.

Components of a pattern

Definition (Borderline tasks)

A task i is a borderline task for the pattern if $t_i + p_i \geq T$.
A borderline task is associated with virtual tasks, a starting task $[t_i, T]$, an ending task $[0, t_i + p_i \mod T]$ and eventually whole period tasks $[0, T]$.

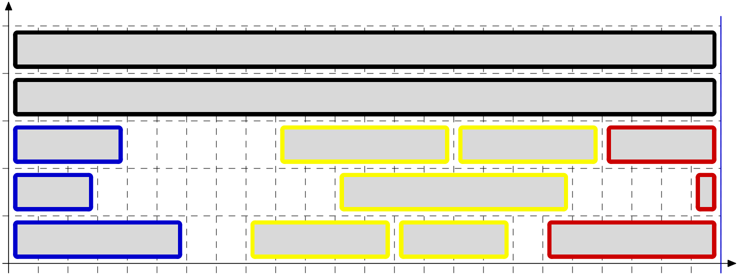
Pattern feasibility

A pattern is feasible if and only if the inner tasks and corresponding virtual tasks of borderline tasks are schedulable.

Components of a pattern

Components of a pattern

- Whole period pieces (in black)
- End of task pieces (in blue)
- Start of task pieces (in red)
- Inner tasks (in yellow)



Whole period tasks

Whole period tasks

A borderline task may be divided in n or $n+1$ whole period pieces. Let δ_i be a binary variable and $a_{i,j}$ be the number of whole period tasks of the borderline task $\langle i, j \rangle$. Then:

$$a_{i,j} + \delta_{i,j} = \lfloor p_{i,j} / T \rfloor$$

Variables

$\beta_{i,j}$ is a binary variable equal to 1 if task $\langle i,j \rangle$ is an inner task and equal to 0 if this is a borderline task.

$$\begin{array}{rrll} & t_{i,j}+p_i & \leq T & +(1-\beta_{i,j})H \\ \beta_{i,j}H+ & t_{i,j}+p_i & \geq T & \end{array}$$

For each task $\langle i,j \rangle$, $start_{i,j}$ and $end_{i,j}$ are the length of the starting and the ending part of a borderline task.

$$\begin{array}{ccccc} 0 & \leq & start_{i,j} & \leq & T \\ 0 & \leq & end_{i,j} & \leq & T \\ p_i(1-\beta_{i,j}) & = & start_{i,j}+a_{i,j}T+end_{i,j} & & \\ start_{i,j} & \leq & T-t_{i,j}+\beta_{i,j}H & & \\ start_{i,j} & \geq & T-t_{i,j}-\beta_{i,j}H & & \end{array}$$

Tasks location

$machine^{start}_{i,j,k}$, $machine^{inner}_{i,j,k}$, $machine^{end}_{i,j,k}$ are binary variables equal to 1 if the starting, inner or ending respectively part of the task $\langle i, j \rangle$ runs on machine k and else equal to 0.
Each piece is at most on one machine:

$$\begin{aligned} \forall i, j, \textstyle\sum_{k=1}^{M} machine^{end}_{i,j,k} &= 1 - \beta_i \\ \forall i, j, \textstyle\sum_{k=1}^{M} machine^{start}_{i,j,k} &= 1 - \beta_i \\ \forall i, j, \textstyle\sum_{k=1}^{M} machine^{inner}_{i,j,k} &= \beta_i \end{aligned}$$

There is at most one starting or ending piece per machine:

$$\begin{aligned} \forall k, \textstyle\sum_{i,j} machine^{end}_{i,j,k} &\leq 1 \\ \forall k, \textstyle\sum_{i,j} machine^{start}_{i,j,k} &\leq 1 \end{aligned}$$

Available machines

We sort the machines so that the whole period tasks are always on the first machines. This means that machines from 1 to $\sum_{i,j} a_{i,j}$ are used by whole period tasks. It remains $M - \sum_{i,j} a_{i,j}$ machines.
Let γ_k be a binary variable such that $\gamma_k = 0$ if the machine k is used by whole period tasks and else $\gamma_k = 1$.

$$\begin{array}{ccc}
\forall k, \gamma_{k+1} & \geq & \gamma_k \\
M & = & \sum_{i,j} a_{i,j} + \sum_k \gamma_k \\
machine^{start}_{i,j,k} & \leq & \gamma_k \\
machine^{end}_{i,j,k} & \leq & \gamma_k \\
machine^{inner}_{i,j,k} & \leq & \gamma_k
\end{array}$$

As we know that $a_{i,j} \geq \lfloor p_i/T \rfloor - 1$, we already have:

$$\forall k \in \{1, \ldots, \sum_{i,j} \lfloor p_i/T \rfloor - 1\}, \gamma_k = 0$$

No overlapping of tasks

$\forall (i_1, j_1) \neq (i_2, j_2),$

$$\begin{array}{ccl} x_{i_1,j_1,i_2,j_2} + x_{i_2,j_2,i_1,j_1} & = & 1 \\ t_{i_2,j_2} & \leq & t_{i_1,j_1} + Hx_{i_1,j_1,i_2,j_2} \\ k, t_{i_1,j_1} + p_{i_1} & \leq & t_{i_2,j_2} + H(3 - x_{i_1,j_1,i_2,j_2} - mach^{interieur}_{i_1,j_1,k} - mach^{interieur}_{i_2,j_2,k}) \end{array}$$

On a given machine all inner and starting (resp. ending and inner) tasks must begin after the end (resp. must end before the starting) task present on that machine it it exists.
$\forall i_1, j_1, i_2, j_2, i_3, j_3, k,$

$$\begin{array}{ccl} end_{i_1,j_1} & \leq & t_{i_2,j_2} + (2 - machine^{end}_{i_1,j_1,k} - machine^{inner}_{i_2,j_2,k})H \\ t_{i_2,j_2} + p_{i_2} + start_{i_3,j_3} & \leq & T + (2 - machine^{start}_{i_3,j_3,k} - machine^{inner}_{i_2,j_2,k})H \\ end_{i_1,j_1} + start_{i_3,j_3} & \leq & T + (2 - machine^{start}_{i_3,j_3,k} - machine^{end}_{i_1,j_1,k})H \end{array}$$

Schedule reconstruction

From the pattern we have to build a periodic schedule.
We have to shift tasks to the next period if it does not respect its release:
Let $t'_{i,j}$ be the real running time, if $t_{i,j} \geq r_{i,j}$ alors $t'_{i,j} = t_{i,j}$, else $t'_{i,j} = t_{i,j} + T$. Thus:

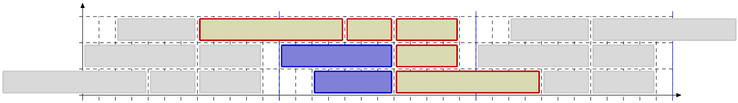
$$\begin{array}{rcl} t'_{i,j} & = & t_{i,j} \quad + \quad \theta_{i,j} T, \theta_{i,j} \in 0, 1 \\ t'_{i,j} & \geq & r_{i,j} \end{array}$$

Presence times are computed from this reconstructed schedule.

Results

Problem	1	2	3	4	5
User	2	3	2	10	2
Jobs	2	4	13	10	6
Machines	4	3	5	3	3
MinMaxPresence	<0.01s	0.01s	0.22s	0.06s	0.17s
WeightedMinMaxPresence	<0.01s	0.01s	0.51s	1.03s	4.51s
MeanPresence	<0.01s	0.01s	0.45s	0.37s	0.11s
WeightedMeanPresence	0.01s	0.01s	0.51s	0.56s	0.19s

Problem 5, WeightedMinMaxPresence:



Summary and Conclusion.

- We defined criterions based on the presence interval of each user.
- We have proposed 2 linear programming models for the scheduling of multi-user periodic arrival of tasks.
  - Transitory model
  - Pattern-based model: much smaller, much faster
- current work: accurate tests of second formulation

- even second formulation will probably not be able to solve medium to large size instances.
- We are now working on heuristics for that problem, based on the Vehicule Routing Problem