# Approximation Algorithms for Multiple Strip Packing

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# Outline

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# Multiple Strip Packing (MSP)

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**Given:**

- a set of $n$ rectangles $r_1, \ldots, r_n$, with heights and widths $\leq 1$,
- $k$ strips of unit width and infinite height.

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**Complexity:**
Strongly NP-hard by reduction to 3-Partition.

# MSP

## Applications

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- Computer grids
- Server consolidations
- Cutting problems

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For a list $L = \{r_1, \ldots, r_n\}$ of rectangles and $k \in \mathbb{N}$ strips $S_1, \ldots, S_k$

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- $A(L) := \max_{i \in \{1,\ldots,k\}} h_i$ denotes the output of an algorithm $A$ for MSP.
- $\mathrm{OPT}(L)$ denotes the optimal value.

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- $(2+\varepsilon)$-Approximation in $\mathcal{O}((n/\varepsilon)^{1/\varepsilon^2})$ (Ye et al. 2009)

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## New results

- Extension of NFDH and FFDH with same ratio
- AFPTAS
- 2-Approximation

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- $k = 2$: PTAS for Rectangle Packing with Area maximization (RPA) by Bansal et al.
- $3 \leq k < c$: Extension of the PTAS for RPA for $k$ bins.
- $k \geq c$: Asymptotic 1.69-approximation for 2-dimensional Bin Packing by Caprara.

# $k = 1$

**Theorem (Steinberg)**

*Let $L = \{r_1, \ldots, r_n\}$ be a set of rectangles with heights and widths $\leq 1$. If $SIZE(L) \leq 1$ then it is possible to pack $L$ into a rectangle $Q = [0,1] \times [0,2]$*

# Rectangle Packing with Area maximization (RPA)

## Formulation of the problem

**Given:**

- a set of $n$ rectangles $r_1, \ldots, r_n$, with heights and widths $\leq 1$,
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## Theorem (Bansal et al.)

*There is a polynomial-time approximation scheme for RPA.*

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2. Scale the heights of the rectangles in $L$ by $1/v$
   $\Rightarrow$ the corresponding packing fits into $[0,1] \times [0,2]$.

# k=2

## Binary Search for $v$

- heights of rectangles $h_i \in \mathbb{Q} \Rightarrow \exists q_i, p_i \in \mathbb{N}$ with $h_i = {}^{p_i}/_{q_i}$
- v is equal to the sum of heights of rectangles in $L$
- for $i \in \{1, \ldots, n\}$ we have $Qh_i \in \mathbb{N}$, where $Q = \prod_{i=1}^{n} q_i$
- $Qh_{\max} \leq Qv \leq Qnh_{\max}$ and $Qv \in \mathbb{N}$

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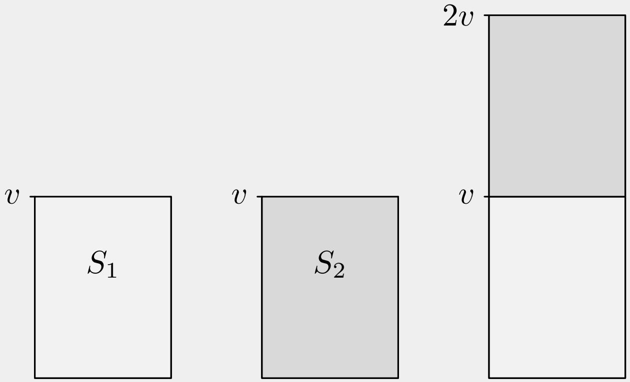
- $Qh_{\max} \leq Qv \leq Qnh_{\max}$ and $Qv \in \mathbb{N}$

$$\Rightarrow \log_2(Qnh_{\max}) = \sum_{i=1}^{n} \log_2(q_i) + \log_2(n) + \log_2(h_{\max}) \leq |L|$$

# $k = 2$

Consider the set of scaled rectangles in $L_v$ as an instance of RPA with $OPT_{RPA}(L_v) = SIZE(L_v) \leq 2$.



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$\Rightarrow$ Find a packing of a subset $L'_v \subset L_v$ with total area $\geq (1-\varepsilon)SIZE(L_v)$ into $[0,1] \times [0,2]$.
Rescaling $L'_v$ gives a packing for $S_1$ with height at most $2v$.

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   Rescaling $L'_v$ gives a packing for $S_1$ with height at most $2v$.
4. Pack remaining items with Steinberg's or Schiermeyer's algorithm into $[0, 1] \times [0, 2]$ and rescale them.
   Possible, since $SIZE(L_v \backslash L'_v) \leq \varepsilon SIZE(L_v) \leq 1/2$.

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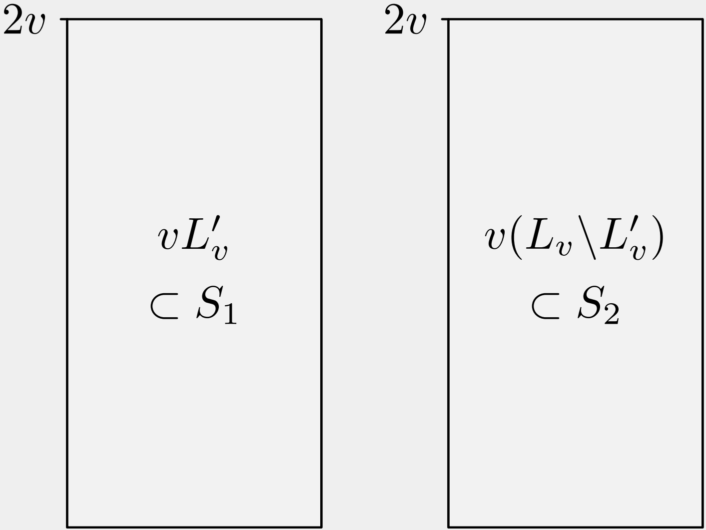

# $3 \leq k < c$

**Theorem (RPA for $k$ bins)**

*For a constant number $k$ of bins, a fixed value $\varepsilon$ and a set of rectangles $L = \{r_1, \dots, r_n\}$ there is a polynomial time algorithm $A_{k,\varepsilon}$ that finds a subset $L' \subset L$ with total area at least $(1 - \varepsilon)SIZE(L)$ and a packing for $L'$ into $k$ bins.*

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### Lemma

*Let $k \geq 3$ and $L$ be an instance of 2-dimensional Bin Packing with total area $SIZE(L) \leq k/4$. There exists a packing of $L$ into $k$ bins.*

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   $\Rightarrow$ the corresponding packing fits into $k$ bins of unit size.

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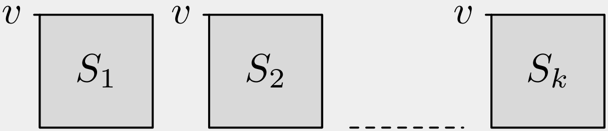
Consider the set of resulting rectangles as an instance of RPA with $k$ bins and $OPT_{RPA} = SIZE(L_v) \leq k$.



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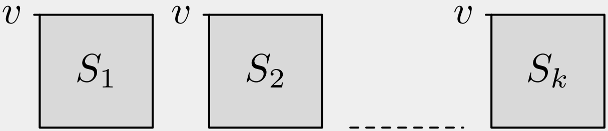


3. Apply the algorithm for RPA with $k$ bins for a **fixed** accuracy $\varepsilon \leq 1/4$ and find a packing for a subset $L'_v \subset L_v$ with total area at least $(1-\varepsilon)SIZE(L_v)$. Rescale the rectangles of $L'_v$ and get $k$ bins of height $v$.

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## Algorithm (Part III)

4 Pack the remaining items into $k$ bins and rescale.
Possible with above Lemma, since $SIZE(L_v \setminus L'_v) \leq \varepsilon SIZE(L_v) \leq k/4$.
This results again in $k$ bins of height at most $v$.

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5 Stack every two bins on top of each other.

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Stack every two bins on each other.

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Stack every two bins on each other.

**Theorem**

*For arbitrary k there is a 2-approxmiation for MSP.*

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2-Approximation

AFPTAS

Conclusion

# AFPTAS

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*There is an AFPTAS for Strip Packing with additive constant* $\mathcal{O}(1/\varepsilon^2)$.

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### Theorem

*For arbitrary $k$ there is an AFPTAS for MSP with additive constant $\mathcal{O}(1/\varepsilon^2)$.*

### Theorem

*For $\varepsilon > 0$ and $k \geq \lceil 128/\varepsilon^3 \rceil$ there is an asymptotic $(1 + \varepsilon)$-approximation for MSP with additive constant $\mathcal{O}(1)$ with running-time polynomial in $n$ and $1/\varepsilon$.*

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- Solve the LP

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where $\alpha_{ij}$ is the number of rectangles of width $w_i$ in $C_j$
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- Construct a feasible solution for $L_{sup}$ by balancing configurations.
- Pack $L_{narrow}$ into the remaining space.

# Balanced configurations

From the LP solution we get a vektor $(x_1, \ldots, x_m)$, $m \leq M$ and a value $h$ with $\frac{\sum_{j=1}^{m} x_j}{k} = h$. We can construct a fractional packing into $k$ strips with height $\leq (1+\varepsilon')h$ and $m' \leq 2M$ different configurations

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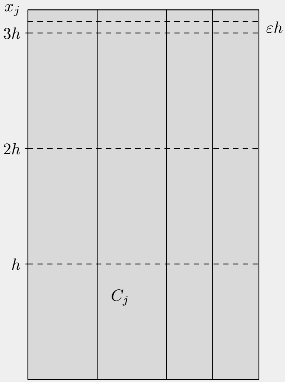

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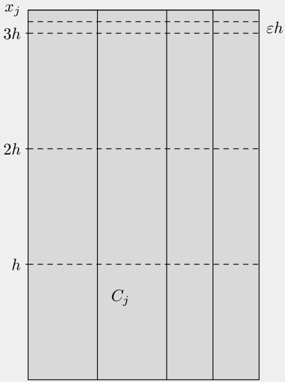


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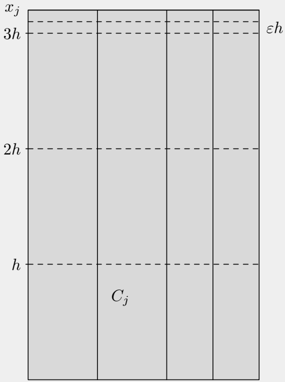


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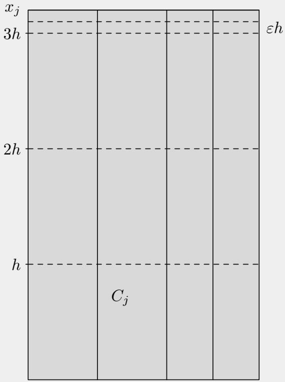
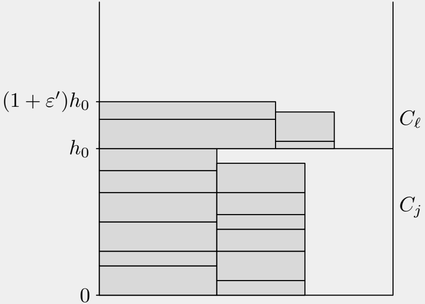



- Fractionally filling the configurations leads to $2M$ non-empty configurations.
- Divide each configuration $C_j$ with $x_j \geq h$ into stripes of height $h$.
- Divide the rest into at most $1/\varepsilon'$ stripes of height $\varepsilon' h$.

# Balanced configurations

- Assign to each strip at most **two** parts of maybe different configurations.



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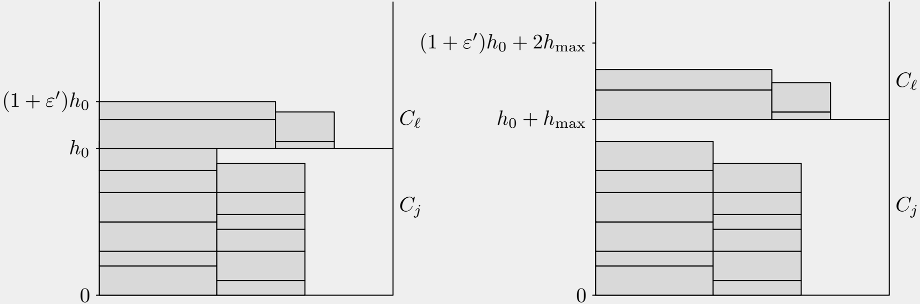

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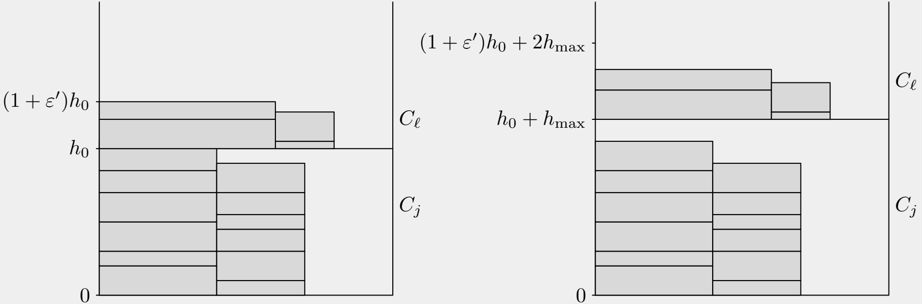


- Pack the narrow items into the remaining space and on top.

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*In the AFPTAS the additive term* $\mathcal{O}(1/\varepsilon^2)$ *improves to* $\mathcal{O}(1)$*, if the number of strips is sufficient large.*

# Conclusion

**Theorem**
*There is an algorithm for MSP with absolute ratio 2.*

**Theorem**
*In the AFPTAS the additive term $\mathcal{O}(1/\varepsilon^2)$ improves to $\mathcal{O}(1)$, if the number of strips is sufficient large.*

**Open questions**

- Improving the running-time
- Different widths for the strips

Thanks for your attention.