The Domino Inequalities: Facets for the symmetric Traveling Salesman Polytope

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Abstract

Adam Letchford defines in [4] the *Domino Parity* inequalities for the Symmetric Traveling Salesman Polytope (STSP) and gives a polynomial algorithm for the separation of such constraints when the support graph is planar, generalizing a result of Fleischer and Tardos [2] for maximally violated comb inequalities. Naddef in [5] gives a set of necessary conditions for such inequalities to be facet defining for the STSP. These conditions lead to the *Domino inequalities* and it is shown in [5] that one does not lose any facet inducing inequality restricting the Domino Parity inequalities to Domino inequalities, except maybe for some very particular case. We prove here that all the domino inequalities are facet inducing for the STSP, settling the conjecture given in [5]. As a by product we will also have a complete proof that the comb inequalities are facet inducing.

1 Introduction

Adam Letchford gives in [4] a polynomial separation algorithm for a class of inequalities which contains the comb inequalities and which he called *Domino Parity* inequalities, in the case the support graph of the solution is planar. This algorithm generalizes the one of Fleischer and Tardos [2] for maximally violated comb inequalities when the support graph is planar. The Domino Parity inequalities are given in a very general form. These inequalities, like the comb inequalities, are defined on a subset of nodes called the *handle* and on an odd number of subsets of nodes called teeth. Unlike the combs, teeth need not be non intersecting and can be quite arbitrary. Boyd, Cockburn and Vella [1] give a first class of facet inducing Domino Parity inequalities. In [5], Naddef gives necessary conditions for Domino Parity inequalities to define facets of the Symmetric Traveling Salesman Polytope and defines the *Domino Inequalities.* It is shown that any facet inducing Domino Parity inequality has a domino inequality which defines the same facet, and therefore one can restrict attention to such domino inequalities. It is also conjectured there that all domino inequalities are facet inducing for the Symmetric Traveling Salesman Polytope.

In the next section we will define the domino inequalities and give a few useful properties. In Section 3 we give a transformation defined by Boyd, Cockburn and Vella [1], which, when performed in a proper way, transforms a domino configuration (or a domino parity configuration) into another equivalent one. Section 4 shows how any domino configuration can be built up from the three tooth comb on six nodes using five operations. In Section 5 we look at some useful properties of tours which satisfy the corresponding domino inequality to equality. We prove in Sections 6.2, 6.3, 7.2 and 7.3 that each of the operations defined in Section 4 when applied to a facet inducing domino configuration yields another such facet inducing configuration. Meanwhile Section 7.1 will have given the main general node lifting theorems from [7] which will be used.

We end this section with some notation. We assume that we deal with a complete graph $K_n = (V, E)$, with $n = |V|$ representing the number of vertices. Let $S \subset V$, then $\delta(S)$ (resp. $\gamma(S)$) represents the set of edges with exactly one endnode in S (resp. both endnodes in S), i.e. $\delta(S) = \{(u, v) \in E : u \in S, v \notin S\}$ (resp. $\gamma(S) = \{(u, v) \in E : u \in S, v \in S\}$). The edge set $\delta(S)$ is in general called the *coboundary* of S (some authors say *cocycle* of S). We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$. For $S \subset V$ and $T \subset V \setminus S$ we denote by $(S : T) = (T : S)$ the set of edges with one endnode in S and the other in T.

We let $\mathcal{H}$ denote the set of all the hamiltonian cycles of K_n and $STSP(n)$ will denote the Symmetric Traveling Salesman Polytope, that is the convex hull of the incidence vectors of all the hamiltonian cycles $\Gamma \in \mathcal{H}$. We will often use *tour* instead of Hamiltonian cycle.

A *spanning closed walk* is a family of edges Γ, a same edge may appear several times in a family, such that, if Γ^* is the set obtained by replacing each edge of Γ by as many copies as the number of times it appears in Γ, the graph (V, Γ^*) is eulerian. All our closed walks will be spanning, therefore from now on closed walk should be understood as spanning closed walk. To a closed walk Γ one can associate a vector $x^\Gamma \in \mathbb{R}^{|E|}$ such that x^Γ_e represents the number of times e is in Γ. We let $GTSP(n)$ represent the convex hull of all these vectors when Γ spans all the closed walks.

Since the content of this paper is not intended for a newcomer in the field of the Traveling Salesman Polytope, we assume the reader familiar with the basics of the Traveling Salesman Polytope and the linear algebra it requires.

2 The Domino inequalities

A set $\mathcal{S}$ of subsets of V is said to be *nested* if any two of its distinct elements either do not intersect or one is contained in the other. Given a nested set $\mathcal{S}$ of subsets, we will call *minimal* those elements of $\mathcal{S}$ which do not contain properly any other element of $\mathcal{S}$, we call *maximal* those which are not properly contained in any other subset $\mathcal{S}$ (a set can be at the same time minimal and maximal). We call *odd* (resp. *even*) a set which, including itself, contains an odd (resp. even) number of subsets of $\mathcal{S}$. Given $S_i \in \mathcal{S}$, we call *maximal* sets of $\mathcal{S}$ contained in S_i, those sets of $\mathcal{S}$ strictly contained in S_i and maximal, relative to set inclusion, with respect to that property.

A nested set $\mathcal{S}$ of subsets of V is said to be *domino covered* if each non minimal set S_i is partitioned into two non empty sets A_i and B_i. We will talk of the *domino* S_i and of its *half dominoes* A_i and B_i. We say that $\mathcal{S}$ is *properly* domino covered if for any non minimal set S_i, for all sets $S_j \subset S_i$ with $S_j \in \mathcal{S}$, one has either $S_j \subset A_i$ or $S_j \subset B_i$, and each of A_i and B_i contains at least one element of $\mathcal{S}$.

A *domino configuration* is defined by a set $H \subset V$, called the handle, and a nested properly domino covered set of so called teeth, $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$, $t \geq 3$ and odd which satisfy Conditions 2.1 to 2.4. When T_j is not minimal let $T_j = A_j \cup B_j$, $A_j \neq \emptyset$, $B_j \neq \emptyset$, $A_j \cap B_j = \emptyset$, and $\alpha_j = |\{i : T_i \subset A_j, T_i \text{ maximal and odd }\}|$ and $\beta_j = |\{i : T_i \subset B_j, T_i \text{ maximal and odd }\}|$. Let α_0 be the number of odd maximal teeth.

$$H \cap T_j \neq \emptyset, \qquad \text{for } j = 1, \ldots, t \tag{2.1}$$

$$T_j \setminus H \neq \emptyset, \qquad \text{for } j = 1, \ldots, t \tag{2.2}$$

$$\alpha_j \geq 2, \ \beta_j \geq 2 \qquad \text{for } T_j \text{ non minimal, } j = 1, \ldots, t, \tag{2.3}$$

$$\alpha_0 \geq 3 \tag{2.4}$$

In other words there is a set H called the handle and a nested properly domino covered set of teeth, such that each tooth intersects properly the handle, there are at least three odd maximal teeth, and each half domino contains at least two odd teeth, maximal with respect to it. The total number of teeth is odd. As a consequence the number of odd maximal teeth is also odd. Figure 1 gives an example. In that figure where a black filled point is present there must be at least one vertex, where a white filled point is present there may or not be some vertices. Note that this example has five odd maximal teeth, among which four are also minimal. The domino partition is shown with dash lines.

Comb configurations are those domino configurations such that all teeth are minimal.

Let $C = (\bigcup_{j=1}^{t} (A_j : B_j)) \setminus \delta(H)$, that is the set of all edges going from a half domino to the other one and which do not cross the border of the

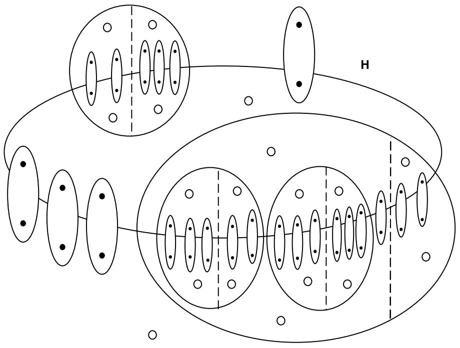

Figure 1: A domino configuration

handle H. The *domino inequality* is:

$$x(\delta(H)) + \sum_{j=1}^{t} x(\delta(T_j)) + 2x(C) \geq 3t + 1 \tag{2.5}$$

Theorem 1 *The domino Inequality 2.5 is valid for STSP(n).*

Proof. Call LHS the left hand side of this inequality, it is easy to check that:

$$LHS \geq \frac{1}{2}\sum_{i=1}^{t} x(\delta(A_i)) + \frac{1}{2}\sum_{i=1}^{t} x(\delta(B_i)) + \frac{1}{2}\sum_{i=1}^{t} x(\delta(T_i)) \geq 3t \tag{2.6}$$

Since LHS is even for all tours, we can raise $3t$ to $3t+1$. □

The reader may have noticed that this is exactly the same proof as the algebraic validity proof for comb inequalities (see [6]).

We say that two vertices u and v are equivalent if they belong to exactly the same sets of the configuration. A domino configuration will be called *simple* if the classes of that equivalence relation all contain at most one node. It is simple and *minimal*, if all nodes belong to minimal teeth. In the example of Figure 1, it is simple and minimal if there is exactly one vertex in each of the positions where a black filled node is present, and no other vertices are present.

3 Some remarks on the domino inequalities

Consider a (not necessarily nested) proper domino covered set of subsets $\mathcal{T}$ and a set H such that Conditions 2.1 and 2.2 hold, and $(A_i : B_i) \cap (A_j : B_j) = \emptyset$, for all $i \neq j$. The Inequality 2.5 still has a meaning, and is valid for $STSP(n)$. This set of inequalities is a superset of the domino inequalities and a subset of the domino parity inequalities. The following theorem applies to such configurations, which, for this section only, we will call *general domino configurations.*

Theorem 2 *[1] In a general domino configuration, consider a non minimal tooth $T = (A \cup B)$, A and B being its two half dominoes. Let $T' = ((V \setminus T) \cup B)$, with half dominoes $V \setminus T$ and B, and $H' = H \Delta A$, the symmetric difference of A and H, then $\mathcal{T}' = \mathcal{T} - T + T'$ and H' define the same inequality as $\mathcal{T}$ and H.*

When two different domino configurations yield the same inequality, we say that they are *equivalent.*

The previous transformation applied to a domino configuration always yields another domino configuration only if it is applied to a maximal tooth T. We will make an extensive use of Theorem 2 because it enables, by recursive application to maximal teeth to make globally maximal all maximal teeth of a given half domino.

The rest of this section can be skipped by the first time reader, it just gives an idea of why one can restrict the study of facet inducing domino parity inequalities to that of domino inequalities.

It will be too time consuming to give the definition of Letchford's Domino Parity inequalities. Nevertheless it is shown in [5] that the only one that could be facet inducing and do not appear in the previous description are those such that either:

- There exist two teeth whose union contains all the vertices of the graph, or
- There is a unique maximal tooth, one of its half domino contains no other tooth and that half domino either does not intersect the handle (Case a) or is entirely contained in it (Case b). In Case (a) there is at least one node outside the maximal tooth and outside the handle. In Case (b) there is at least a node in the handle not in the maximal tooth, or
- There is a unique odd maximal tooth, tooth which is necessarily not minimal, and at least two even maximal teeth. No node is not contained in any tooth. Other than that all the conditions of the domino configuration are satisfied.

For the third case we conjecture that the corresponding inequality is not facet inducing. This is the only open problem relative to the facet inducing domino parity inequalities which is not settled in this paper. Note using the transformation of Theorem 2, it may take the form of a unique half domino contains a unique odd maximal tooth which cannot be also minimal. That half domino contains at least two other teeth and no node outside any tooth.

Our definition of *proper* avoids the second case. When it occurs one can find a nested set and a modified handle, that satisfy our requirements and defines the same inequality (see [5]). Figure 2 gives an example of one such configuration which turns out to be equivalent to a comb.

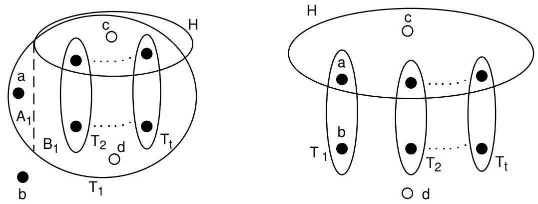

Figure 2: A degenerate domino parity configuration and an equivalent comb

Figure 3(a) shows sets, handle and teeth with proper domino partitions which satisfy all the conditions of the domino configuration, except for the nestedness and there are two teeth the union of which is V. Applying the transformation with $T = T_1$ yields the domino configuration of Figure 3(b). The two handles H and H' differ in the two grey vertices. The two grey vertices of the second are the grey vertices which do not belong to the first one.

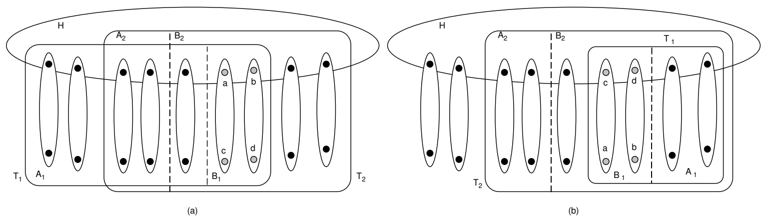

Figure 3: Pathological domino parity & equivalent domino configuration

4 Construction of domino configurations from minimal simple three tooth combs

A non minimal tooth will be called *near minimal* if it contains only minimal teeth. A near minimal tooth will be called a *fundamental* tooth if either it is odd and contains two or three minimal teeth in each of its half dominoes, or it is even and its half dominoes contain respectively exactly two and three minimal teeth.

Let $\mathcal{C}$ be the equivalence classes for the relation $u\mathcal{R}v$ if and only if u and v belong exactly to the same sets of the domino configuration.

Consider the following *destructive* operations, where the two first ones are carried out at once, the following ones recursively after that:

1. Shrink each class of $\mathcal{C}$ into a single node.

2. Remove all nodes which do not belong to a minimal tooth.

3. In a near minimal tooth which is not fundamental, remove two minimal teeth from the *same* half domino, if it leaves at least two minimal teeth in it.

4. Remove two minimal teeth which are also maximal (that is outside any other tooth) if it leaves at least three odd maximal teeth.

5. Remove an even fundamental tooth.

6. Replace an odd fundamental tooth by a minimal tooth.

After Operation 1, one obtains what is now commonly called a *simple* configuration. After Operation 2, one obtains a minimal simple configuration. In the preceding drawings it corresponds to no nodes in the position of white filled points and exactly one node in the position of black filled points.

It is quite obvious that when performed all the way, one ends with a simple three tooth comb of K_6. As a consequence, reversing the previous operations one can build any domino configuration from a simple three tooth comb by the following operations:

Domino construction procedure: Perform as long as necessary any of the following operations:

1. Add two minimal teeth inside a same half domino.

2. Add two minimal teeth outside any other tooth (augmenting the number of maximal odd teeth).

3. Add an fundamental even tooth inside some half domino.

4. Add an even tooth outside any other tooth (augmenting the number of maximal teeth).

5. Replace a minimal tooth by a fundamental odd tooth.

And after that use what is commonly called *1-node lifting* procedures to restore a node in the "white" positions, where necessary, and then use what is commonly called a *0-node lifting* to replace each node by a clique of desired size.

Using Theorem 2, recursively, starting with the maximal tooth containing the half domino in which we want to do some transformation, one can assume that all the operations are carried out outside any tooth. This is what we will assume from now on, and therefore only Operations 2, 4 and 5 will have to be considered.

5 Properties of tight tours for a domino inequality

A tour $\Gamma \in \mathcal{H}$ is said to be *tight* for a domino inequality if its incidence vector satisfies that inequality to equality. A subset S is *tight* for a given tour $\Gamma \in \mathcal{H}$ if $|\Gamma \cap \delta(S)| = 2$. It is quite important to understand what the tight tours look like.

We call *penalized* an edge of $C = (\bigcup_{j=1}^{t}(A_j : B_j)) \setminus \delta(H)$.

For notational convenience, for a minimal tooth T_i, we let $A_i = T_i \cap H$ and $B_i = T_i \setminus H$. Note that it does not affect the set of penalized edges C since all edges of $(A_i : B_i)$ belong the $\delta(H)$.

The proof of the three following lemmas are a direct corollary of the proof of validity of domino inequalities. In all cases, tours can be replaced by spanning closed walks, and the lemmas remain true.

Lemma 3 *Let $cx \geq c_0$ be a domino inequality. Let Γ be any tight tour for this inequality. At most one of the sets A_i, B_i and T_i, $i = 1, \ldots, t$ is not tight for Γ.*

Lemma 4 *Let $cx \geq c_0$ be a domino inequality. Let Γ be any tight tour for this inequality. Then Γ contains at most one penalized edge. In the case Γ contains a penalized edge, all the sets A_i, B_i and T_i, $i = 1, \ldots, t$ are tight for Γ. Moreover the $t-1$ edges of $\Gamma \cap \delta(H)$ are in one to one correspondance with the t teeth.*

Lemma 5 *Let Γ be a tight tour such that tooth T_j is not tight. There is a one to one correspondence between the edges of $\delta(H) \cap \Gamma$ and the teeth other than T_j. In that correspondance, the edge corresponding to T_i belongs to $(A_i : B_i)$.*

To summarize, given a domino inequality, tight tours Γ fall in one of the following categories:

1. All teeth are tight and Γ contains no penalized edge. Then $|\delta(H) \cap \Gamma| = t+1$. Let $e^* \in \delta(H) \cap \Gamma$ be such that $|\Gamma \cap (A_j : B_j) \cap (\delta(H) \setminus \{e^*\})| = 1$ for all $j = 1, \ldots, t$. The edge e^* will be called the *joker*. The joker may be any edge, but if an edge of $\delta(H)$ belongs to no $(A_j : B_j)$, the only way it can belong to a tight tour is by being the joker edge.

2. All teeth are tight and Γ contains one penalized edge $e \in (A_j : B_j)$. Then $|\delta(H) \cap \Gamma| = t - 1$ and $|\Gamma \cap ((A_i : B_i) \cap \delta(H))| = 1$ for all $i = 1, \ldots, t$, $i \neq j$, and therefore there is no joker edge.

3. All teeth, except one, say T_j, are tight. Then Γ contains no penalized edge, $|\delta(H) \cap \Gamma| = t - 1$ and $|\Gamma \cap ((A_i : B_i) \cap \delta(H))| = 1$ for all $i = 1, \ldots, t$, $i \neq j$, and therefore there is no joker edge.

Let Γ be a tight tour, for which the non minimal tooth T is tight. Then the following may happen (recall that if a penalized edge is used, no joker edge can be used and conversely, if a joker edge is used, no penalized edge can be used):

1. T is odd, and no joker edge nor penalized edge is used inside T, then Γ enters and exits T on opposite sides of the handle. If a joker edge or a penalized edge with both ends in T is used, then Γ enters and exits T on the same side of the handle.

2. T is even, and no joker edge nor penalized edge is used inside T, then Γ enters and exits T on the same side of the handle. If a joker edge or a penalized edge with both ends in T is used, then Γ enters and exits T on opposite sides of the handle.

We will refer to a *proper visit* of a tooth T^*, as being a path that visits every node inside T^* without using a penalized edge nor a joker edge and such that all teeth inside T^*, including T^*, is tight. Note that a proper visit enters and exits an odd tooth on opposite sides of the handle and on the same side for an even one. Also note that in a proper visit, every nodes of a same class are visited consecutively, therefore from now on we can assume that the domino configuration is simple, but not necessarily minimal.

The nodes $a_i, \overline{a}_i, b_i, \overline{b}_i$ will be called *free nodes*.

A *standard path* is a path that can be completed to a tight tour and which contains no penalized nor joker edge. Two teeth are said to be at the *same level* if they are either maximal or in the same half domino of some tooth T_j and maximal relative to that tooth. A standard path, when exiting a tooth, visits, if necessary, the free node of the half domino on the same side and then enters another unvisited tooth of the same level which it visits properly. If no such tooth exists and the other half domino has not yet been visited, it crosses over to that half domino using an edge of $\delta(H)$. It does not matter whether one crosses over to a free node or not. The proof of the next lemma will show more in detail how such a standard path can be built.

Lemma 6 *Let T_j be a tooth of a domino configuration and $a \in T_j$ one of its nodes. There exists a proper visit of T_j that has "a" as one of its ends.*

Proof. It is obviously true if T_j is a minimal tooth. Assume there is a tooth of a domino configuration for which this is not true, consider one which is minimal for this property, that is the property is true for all teeth inside it. We now build a proper visit of T_j, contradicting that no such visit exists. We assume $a \in A_j \cap H$. Let us first assume that the configuration is also minimal. Start at a and properly visit the maximal tooth (relative to T_j) that contains a. Then go to a yet unvisited maximal tooth of the same half domino without crossing the border of the handle, properly visit it and repeat the process until all maximal teeth of that half domino have been visited. Then cross over to the other half domino using an edge of $\delta(H)$, visit properly one after the others all the maximal teeth included in that half domino, going from one tooth to the other without using an edge of $\delta(H)$. If the configuration is not minimal, there may be nodes, including a itself, which are not contained in any maximal tooth relative to T_j. If a is such a node, then start at a and enter any maximal tooth (relative to A_j) remaining in the handle, and proceed as just described. Let Γ be the path obtained. If it is not yet a proper visit of T_j, it is because some free nodes exist, $a_j \in A_j \cap H, \overline{a}_j \in A_j \setminus H, b_j \in B_j \cap H, \overline{b}_j \in B_j \setminus H\}$, and have not yet been visited. By the fact that there are at least two odd teeth in each half domino, the path Γ contains an edge in each of the sets $\gamma(A_j \cap H)$, $\gamma(A_j \setminus H)$,$\gamma(B_j \cap H)$ and $\gamma(B_j \setminus H)$. One can therefore insert the missing nodes in Γ to obtain a proper visit of T_j. This proper visit contradicts our initial assumption that none existed. □

Lemma 7 *Let T_j be a non minimal tooth of a domino configuration. Let $a \in A_j$ and $b \in B_j$ be two nodes of T_j in different half dominoes. There always exists a tight tour Γ such that T_j and all teeth contained in it are tight and Γ enters and exits T_j in a and b respectively.*

Proof. Note that depending on the parity of T_j and the positions of nodes a and b, Γ may or not induce a proper visit of T_j. Wlog we assume $a \in H$. We show how to build such a tour. Build a standard path not containing b, starting at a and visiting all the free nodes, except eventually b, and all the teeth except the one eventually containing b. Say it ends in c. If b is contained in some T_j-maximal tooth, let d be the other end of the proper visit of that tooth starting at b, else (b is a free node) let $d = b$. Link c and d by an edge. That edge may or not be in $\delta(H)$. If it is, we have then used up the joker edge. To finish building the tight tour Γ, from b build a standard path that visits all the nodes outside T_j and finally link the last node of that path to a. This last edge will be the joker edge if it has not yet been used inside T_j. The parity of the number of teeth will never oblige us to use two joker edges, since all teeth are tight and no penalized edge is used. □

Lemma 8 *Consider a simple domino configuration such that $H \setminus \cup_{i=1}^{t} T_i = \emptyset$. Let $e = (u, v)$ and $e' = (u', v')$ be two edges in $\gamma(H)$, with nodes u, u' belonging to different maximal teeth say T and T', and v, v' to two different half dominoes of a maximal tooth T^*, $T^* \neq T$ and $T^* \neq T'$. There exists a tight tour Γ such that $e \in \Gamma$ and $e' \in \Gamma$.*

Proof. Consider a path through T^* linking v to v' as built in the proof of Lemma 7. It may or not contain a joker edge depending on the parity of tooth T^*. Extend that path on both ends by the edges e and e'. We now have a path $\Gamma_{uu'}$ linking u and u'. Properly visit T and T' starting at u and u' respectively, say we end in w and w', respectively. Adding these two paths to $\Gamma_{uu'}$, we obtain a path between w and w', say $\Gamma_{ww'}$, which properly visits T and T', and visits T^*, but not necessarily properly. If there are no other maximal teeth then w and w' are not in the handle (because T and T' are then odd), let $\Gamma = \Gamma_{ww'} + \{(w, w')\}$ or $\Gamma = \Gamma_{ww'} + \{(w, \overline{a}_0), (\overline{a}_0, w')\}$ if there exists a node $\overline{a}_0 \in V \setminus (H \cup_{i=1}^t T_i)$. Else from w build a standard path visiting all the unvisited nodes. Connect by an edge the last node of that path to w'. If no joker edge has been used so far, this edge will be automatically be in $\delta(H)$, else it will not be, by parity on the total number of teeth. The tour obtained is tight for the domino inequality since all teeth are tight and a unique joker edge has been used. □

Lemma 9 *Consider a minimal simple domino configuration. Let* $e = (u, v)$ *and* $e' = (u', v')$ *be two edges in* $\gamma(H)$*, with nodes* u, u' *belonging to different maximal teeth say* T *and* T'*, and* v, v' *to two different minimal teeth which are also maximal, different from* T *and* T'*. There exists a tight tour* Γ *such that* $e \in \Gamma$ *and* $e' \in \Gamma$*.*

Proof. Let $\{v, w\}$ be one of the minimal and maximal tooth and $\{v', w'\}$ be the other. Let $\Gamma_{uu'} = \{e, (v, w), (w, w'), (w', v'), e'\}$, one can build from there on a tight tour just like in the proof of Lemma 8. □

Lemma 10 *Consider a simple domino configuration such that* $H \setminus \cup_{i=1}^t T_i = \emptyset$*. Let* $T = \{a, b\}$ *be a minimal and maximal tooth, and let* $T' \neq T''$ *be two other maximal teeth. Let* $u' \in T' \cap H$ *and* $u'' \in T'' \cap H$*. There exists a tight tour containing edges* (a, u') *and* (a, u'')*.*

Proof. Let $\Gamma_{u'v'}$ and $\Gamma_{u''v''}$ be proper visits of T' and T'' linking from u' to some $v' \in T'$ and u'' to some $v'' \in T''$, respectively. Consider the path $\Gamma_{v'v''} = \Gamma_{u'v'} + \{(a, u'), (a, u'')\} + \Gamma_{u''v''}$. From v'' build a standard path $\Gamma_{v''w}$ visiting all the yet unvisited nodes including node b (which is always possible because either T'' is odd and therefore $v'' \notin H$ or it is even and therefore there exists another odd maximal tooth the proper visit of which will bring us out of the handle). Let w be the other extremity of this standard path (w may be b or $\overline{a}_0$). If both teeth T' and T'' are of same parity, all three nodes v', v'' and w are on the same side of the handle, therefore $(w, v') \notin \delta(H)$. Assume T'' is odd and T' even, then, by the parity of the total number of teeth, w and v' both belong to the handle and again $(w, v') \notin \delta(H)$. The tour $\Gamma_{v'v''} + \Gamma_{v''w} + \{(w, v'\}$ is tight and contains the desired edges. □

Lemma 11 *Consider a not necessarily simple domino configuration and* T *one of its minimal and maximal tooth such that* $T \cap H = \{a\}$*. Let* $u \in V \setminus T$ *and* $v \in T \setminus H$*, there exists a tight tour that contains both edges* (u, a) *and* (a, v)*.*

Proof. Let $\Gamma_{vv'}$ be a hamiltonian path of $T \setminus H$ linking v to some $v' \in T \setminus H$. Consider the path $\Gamma_{uv'} = \{(u,a),(a,v)\} + \Gamma_{vv'}$. Consider a standard path Γ_{uw} that visits all the yet unvisited nodes. By a parity argument often used so far, u and w are on the same side of the handle. If $u \in H$, no joker edges has been used so far, so the edges (v', w) will be that edge, else if $u \notin H$, the edge (a, u) is the joker, but then $(v', w) \notin \delta(H)$ therefore in all cases the tour $\Gamma_{uv'} + \Gamma + \{(v', w)\}$ is tight and contains the desired edges. □

6 Facet preserving operations

6.1 A facet inducing proof technique for $STSP(n)$

Assume one wants to prove that the *valid* inequality $fx \geq f_0$ is facet inducing. Since it is valid it induces a face, that face is contained in some facet. Let $gx \geq g_0$ be a linear inequality which describes that facet. The problem is that there is an infinite number of linear inequalities that define the same facet. One way of getting around this is to set the value of g_e for $e \in F^*$, where F^* is a set of edges such that each component of the graph (V, F^*) has a unique cycle, cycle which is of odd length. Then if $fx \geq f_0$ is facet inducing one should end with $g_e = f_e$ for all $e \in E$ and $g_0 = f_0$. See Grötschel and Pulleyblank [3] for more details. We will always use F^* as the set of edges of a spanning tree plus an edge closing an odd cycle.

6.2 Adding two minimal teeth or an even fundamental tooth inside a half domino

In Section 7.2 we will show that if a minimal simple domino configuration is facet inducing, then so is the simple inequality obtained by adding nodes in half dominoes not contained in any of the teeth the concerned half dominoes contain (free nodes). For this section we only need the result for a unique such node addition, one in the half domino in which one wants to add two minimal teeth or an even fundamental tooth. This unique node may be inside or outside the handle.

The aim of this section is to show that if a simple domino configuration is facet inducing, then so is the configuration obtained by either adding two minimal teeth or an even fundamental tooth either inside a half domino or outside any maximal tooth. By adding we mean that the new teeth are additional maximal teeth contained in that half domino, and therefore not contained in any of the other teeth that half domino contains. As already mentioned, using recursively Theorem 2, starting with the maximal tooth containing A_i, one may assume that the addition is performed outside any tooth. Therefore the node concerned, in what follows, with the 1-node lifting is either in the handle or outside the handle, but in all cases outside all teeth.

Theorem 12 *Let $\mathcal{D}$ be a facet inducing minimal simple domino configuration. Let $A = A_i$ for some tooth T_i, or $A = V \setminus \cup_{i=1}^t T_i$. Adding two minimal teeth in A yields a facet inducing minimal simple domino configuration $\mathcal{D}^*$.*

Proof. We assume that $A = V \setminus \cup_{i=1}^t T_i$. We add teeth $T_{t+1} = \{a, b\}$ and $T_{t+2} = \{c, d\}$ (see Figure 4).

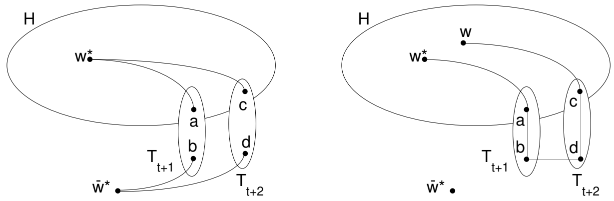

Figure 4: Adding two minimal teeth

Assume that the corresponding domino inequality $d^*x \geq d_0^*$ is not facet inducing. The face it induces is contained in the facet defined by $gx \geq g_0$. Assume $g_e = d_e^*$ for all $e \in F^*$, with F^* such that for given $w^* \in (H \setminus (T_{t+1} \cup T_{t+2})$ and $\overline{w}^* \notin (H \cup T_{t+1} \cup T_{t+2})$, $\{(w^*, a), (w^*, c), (\overline{w}^*, b), (\overline{w}^*, d)\} \subset F^*$, and except for these four edges F^* is a spanning subgraph of $V \setminus (T_{t+1} \cup T_{t+2})$ with a unique cycle, cycle which is of odd length. Let $F = F^* \setminus \{(w^*, a), (w^*, c), (\overline{w}^*, b), (\overline{w}^*, d)\}$.

We first prove that $g_{wa} = g_{wc}$ for all $w \in H \setminus \{a, c\}$. Let w be any node not in the same maximal tooth as w^*. By Lemma 9, there is a d^*-tight tour Γ which contains edges (w^*, a) and (w, c). This tour must also be tight for $gx \geq g_0$. The tour $\Gamma' = \Gamma \setminus \{(w^*, a), (w, c)\} + \{(w^*, c), (w, a)\}$, is also d^*-tight and therefore must also be g-tight, so we get $g_{w^*a} + g_{wc} = g_{w^*c} + g_{wa}$, but since $g_{w^*a} = d^*_{w^*a} = d^*_{w^*c} = g_{w^*c}$ we get $g_{wa} = g_{wc}$, for all w in H and not in the same maximal tooth as w^*. Let v^* be a node of the handle not in the same maximal tooth as w^*. We already know that $g_{v^*a} = g_{v^*c}$, letting v^* play the role of w^* in the previous lines, one shows that $g_{wa} = g_{wc}$ for all the nodes of the handle for which we did not have yet the result. Since the handle and its complement play the same role, one gets, in a similar manner, that $g_{\overline{w}b} = g_{\overline{w}d}$ for all $\overline{w} \notin H \cup T_{t+1} \cup T_{t+2}$.

Now let $\overline{w} \notin H \cup T_{t+1} \cup T_{t+2}$, and consider the path made up by the set of edges $\{(\overline{w}, a), (a, b), (b, d), (d, c)\}$, it can be completed into a d^*-tight tour Γ, since it has its two ends on opposite sides of the handle and there remains an odd number of odd maximal teeth to visit (note that the joker has been already used up). Let $w \neq d$ be such that $(w, c) \in \Gamma$, necessarily $w \in H$ since the joker edge has already been used up. The tour $\Gamma' = \Gamma \setminus \{(\overline{w}, a), (w, c)\} + \{(\overline{w}, c), (w, a)\}$ is d^*-tight and therefore also g-tight and since $g_{wa} = g_{wc}$ one has $g_{\overline{w}a} = g_{\overline{w}c}$ for all $\overline{w} \notin (H \cup T_{t+1} \cup T_{t+2})$. By inverting the roles of H and $V \setminus H$, one obtains similarly: $g_{wb} = g_{wd}$ for all

$w \in H \setminus (T_{t+1} \cup T_{t+2})$.

Let us go back to the domino configuration $\mathcal{D}$. Let $\mathcal{D}^+$ be the domino configuration obtained from $\mathcal{D}$ by adding a single node, say h, in the handle outside all teeth. By Theorem 17, $\mathcal{D}^+$ is facet inducing, and so is the inequality obtained by adding to it the degree equality $x(\delta(h)) = 2$. Let $F^+ = F + \{(w^*, h)\}$. Let $\mathcal{B}$ be the set of tours that enable to prove that this inequality is facet inducing, using F^+ as the set of edges with fixed value.

Similarly let $\overline{\mathcal{D}^+}$ be the domino configuration obtained from $\mathcal{D}$ by adding a single node, say $\overline{h}$, outside the handle and outside all teeth. By Theorem 17, $\overline{\mathcal{D}^+}$ is facet inducing. This is also the case for the inequality obtained by adding to it the degree equality $x(\delta(\overline{h})) = 2$. Let $\overline{F}^+ = F + \{(\overline{w}^*, \overline{h})\}$. Let $\overline{\mathcal{B}}$ be the set of tours that enable to prove that last inequality is facet inducing, using $\overline{F}^+$ as the set of edges with fixed value.

Let $\Gamma \in \mathcal{B}$, we extend it to a tight tour Γ^* of $\mathcal{D}^*$ in the following way: Let h^+ and h^- be the nodes adjacent to h on Γ. Let $\Gamma^* = \Gamma - \{(h^+, h), (h, h^-)\} + \{((h^+, a), (a, b), (b, d), (d, c), (c, h^-)\}$. Let $\mathcal{B}^*$ be the set of tours obtained this way when $\Gamma \in \mathcal{B}$.

Similarly define $\overline{\mathcal{B}^*}$, by extending the tours in $\overline{\mathcal{B}}$ to tours of of $\mathcal{D}^*$.

The tours in $\mathcal{B}^*$ (together with F^+) enable to show that $g_e = d_e$ for all $e \in E \setminus ((\gamma(T_{t+1} \cup T_{t+2}) \cup (\{b, d\} : V \setminus \{a, b, c, d\}))$, that is all edges except those linking two nodes of the added teeth or linking one of the nodes b and d to a node outside the added teeth.

The tours in $\overline{\mathcal{B}^*}$ (together with $\overline{F}^+$) enable the same result except for the roles b and d which are taken by a and c.

Therefore the only edges e for which we do not know the coefficient g_e are those of $\gamma(\{a, b, c, d\})$.

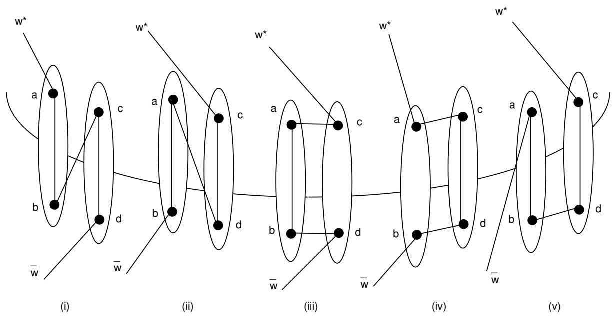

Figure 5: Traces of tight tours

All the set of edges shown in Figure 5 can be completed to a tight tour by the same set of edges. They show that:

- $g_{ad} = g_{bc}$ by comparing (i) and (ii)
- $g_{ab} = g_{cd}$ by comparing (iii) and (iv)

- $g_{ad} = g_{bd} + 1$ by comparing (ii) and (v), since $g_{\overline{w},a} = d_{\overline{w},a} = d_{\overline{w},b} + 1 = g_{\overline{w},b} + 1$ and $g_{w^*,a} = g_{w^*,c}$.
- $g_{ac} = g_{ab} + 1$ by comparing (iv) and (v), since $g_{\overline{w},a} = d_{\overline{w},a} = d_{\overline{w},b} + 1 = g_{\overline{w},b} + 1$ and $g_{w^*,a} = g_{w^*,c}$.

Therefore everything can be derived from the value of $g_{ab} = \alpha$.

Consider the tight tour Γ_1 built as follows. Starting the tour by visiting b, a, d, c, in this order, then from c visit properly a first odd tooth, exiting that tooth in node u, enter the next odd tooth in say node v, visit next properly all the odd maximal teeth, and end by visiting properly all the even maximal ones (if necessary) and then closing the cycle by returning to node b. Let $\Gamma_2 = \Gamma_1 \setminus \{(a,d),(u,v)\} \cup \{(a,u),(d,v)\}$. Figures 6(a) and 6(b) sketch these two tours, where teeth with an "O" are odd and those with an "E" are even. Both tours are tight because all teeth are tight, no penalized edge used, and exactly one joker edge is used.

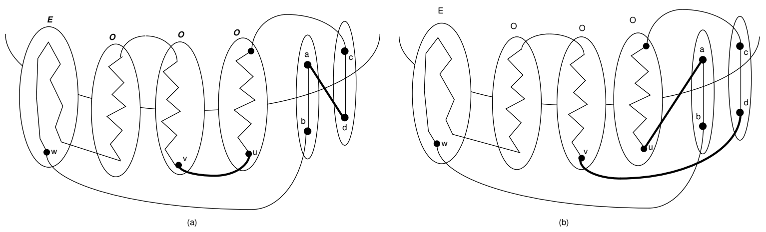

Figure 6: Tight tours

By the definition of the coefficients of the non penalized edges of a domino inequality which are equal to the number of coboundaries they belong to, one has $g_{uv} = d_{uv} = d_{vd} + d_{ua} - 3$ which yields $g_{ad} = 3$ and therefore $\alpha = 1$, which ends the proof. □

Theorem 13 *Let $\mathcal{D}$ be a facet inducing minimal simple domino configuration. Let $A = A_i$ for some tooth T_i, or $A = V \setminus \cup_{i=1}^{t} T_i$. Adding one fundamental even tooth in A yields a facet inducing minimal simple domino configuration $\mathcal{D}^*$.*

Proof. The proof is very similar to the previous case. Wlog we assume again that $A = V \setminus \cup_{i=1}^{t} T_i$. We add the even tooth T_{t+6}, which contains the minimal teeth $T_{t+i} = \{a_i, b_i\}$ for $i = 1, \ldots, 5$. We assume the three first minimal teeth are in A_{t+6} (see Figure 7)

Assume that the corresponding domino inequality $d^* x \geq d_0^*$ is not facet inducing. The face it induces is contained in the facet defined by $gx \geq g_0$. Assume $g_e = d_e^*$ for all $e \in F^*$, with F^* such that for given $w^* \in H \setminus T_{t+6}$ and $\overline{w}^* \notin H \cup T_{t+6}$, $(w^*, a_i) \in F^*$, and $(\overline{w}^*, b_i) \in F^*$ for $i = 1, \ldots, 5$. Let

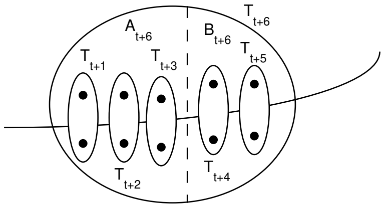

Figure 7: Adding an even fundamental tooth

$F = F^* \setminus \cup_{i=1}^5 \{(w^*, a_i), (\overline{w}^*, b_i)\}$, one requires that F induces a spanning subgraph of $V \setminus T_{t+6}$ with a unique cycle and that cycle is of odd length.

We first prove that for any $w \in H \setminus T_{t+6}$, the value g_{w,a_i} only depends on w and not on $i = 1, \ldots, 5$. Let w be any node not in the same maximal tooth as w^*. By Lemma 8, there is a d^*-tight tour Γ which contains edges (w^*, a_i) and (w, a_j), with $i \in \{1,2,3\}$ and $j \in \{4,5\}$. This tour must also be tight for $gx \geq g_0$. The tour $\Gamma' = \Gamma \setminus \{(w^*, a_i), (w, a_j)\} \cup \{(w^*, a_j), (w, a_i)\}$, is also d^*-tight and therefore must also be g-tight, so we get $g_{w^*a} + g_{wc} = g_{w^*c} + g_{wa}$, but since $g_{w^*a_i} = d^*_{w^*a_i} = d^*_{w^*a_j} = g_{w^*a_j}$ we get $g_{w,a_i} = g_{w,a_j}$, with $i \in \{1,2,3\}$ and $j \in \{4,5\}$ and for all w in H and not in the same maximal tooth as w^*. It is trivial, by fixing i or j and varying the other index, to get that $g_{w,a_i} = g_{w,a_j}$ for all $i \neq j$, for all w in H and not in the same maximal tooth as w^*. Let v^* be a node of the handle not in the same maximal tooth as w^*. We already know that $g_{v^*a_i} = g_{v^*a_j}$, $i \neq j$, and letting v^* play the role of w^* in the previous lines, one shows that $g_{w,a_i} = g_{w,a_j}$ for all the nodes w of the handle for which we did not have yet the result. Since the handle and its complement play the same role, one gets similarly, by inverting the roles of H and $V \setminus H$, that $g_{\overline{w}b} = g_{\overline{w}d}$ for all $\overline{w} \notin H \cup T_{t+1} \cup T_{t+2}$.

Now let $\overline{w} \notin H \cup T_{t+6}$, and consider the path Π made up by the set of edges: $\Pi = \{(a_1,b_1),(b_1,b_2),(b_2,a_2),(a_2,a_3),(a_3,b_3),(b_3,a_4),(a_4,b_4),(b_4,b_5),(b_5,a_5)\}$, then $\Pi + (\overline{w}, a_1)$, can be completed into a d^*-tight tour Γ, since it has its two ends on opposite sides of the handle and there remains an odd number of teeth to traverse. Let $w \neq b_5$ be such that $(w, a_5) \in \Gamma$, necessarily $w \in H$ since the joker edge has already been used up. The tour $\Gamma' = \Gamma \setminus \{(\overline{w}, a_1), (w, a_5)\} + \{(\overline{w}, a_5), (w, a_1)\}$ is d^*-tight and therefore also g-tight and since $g_{w,a_1} = g_{w,a_5}$ one has $g_{\overline{w},a_1} = g_{\overline{w},a_5}$ for all $\overline{w} \notin (H \cup T_{t+1} \cup T_{t+2})$. One can prove similarly that $g_{w,a_i} = g_{w,a_5}$ for all $i = 1,2,3$, and then finally that $g_{w,a_1} = g_{w,a_4}$, which yields that g_{w,a_i} has the same value for all $i \in \{1,2,3,4,5\}$ and all $w \notin T_{t+6}$ (combining with the previous result for $w \in H$). By inverting the roles of H and $V \setminus H$ one obtains similarly that g_{w,b_i} has the same value for all $i \in \{1,2,3,4,5\}$ and all $w \notin T_{t+6}$.

Let us go back to the domino configuration $\mathcal{D}$. Let $\mathcal{D}^+$, F^+, $\mathcal{B}$, $\overline{\mathcal{D}^+}$, $\overline{F}^+$ and $\overline{\mathcal{B}}$ be as in the proof of Theorem 12. By Theorem 17, $\mathcal{D}^+$ and $\overline{\mathcal{D}^+}$ are facet inducing. Let $\Gamma \in \mathcal{B}$, we extend it to a tight tour Γ^* of $\mathcal{D}^*$

in the following way: Let h^+ and h^- be the nodes adjacent to h on Γ. Let $\Gamma^* = \Gamma - \{(h^+, h), (h, h^-)\} + \{((h^+, a_1), \Pi, (a_5, h^-)\}$, where Π is the previously defined path across T_{t+6}. Let $\mathcal{B}^*$ be the set of tours obtained this way when $\Gamma \in \mathcal{B}$.

Similarly define $\overline{\mathcal{B}^*}$, by extending the tours in $\overline{\mathcal{B}}$ to tours of of $\mathcal{D}^*$. The tours in $\mathcal{B}^*$ (together with F^+) enable to show that $g_e = d_e$ for all $e \in E$ except:

- edges with both ends in T_6.
- edges (w, b_i) with $w \in H$, $i = 1, \dots 5$.
- edges $(\overline{w}, b_i)$ with $\overline{w} \notin H$, $i = 1, \dots 5$.

Note that for $w \in H$ one only gets the values of g_{w,a_i} for $i = 1, 5$, but since we have already proved that the value is the same for the other indices, we are done for these edges. Note also that one gets the value of $g_{\overline{w},a_i}$, for $i = 1, 5$ and $\overline{w} \notin H$, the value for the others derive from the equality of all these coefficients.

The tours in $\overline{\mathcal{B}^*}$ (together with $\overline{F}^+$) show that $g_e = d_e$, mostly for edges for which one already knows the result, but also for:

- edges $(\overline{w}, b_i)$ with $\overline{w} \notin H$, $i = 1$ and 5, and therefore for all i.
- edges w, b_i with $w \in H$, $i = 1$ and 5, and therefore for all i.

So we are left with the problem of determining the coefficients of the edges with both ends in T_6 (i.e. of $\gamma(T_6)$).

In all the figures, the nodes u and v are assumed to be in or out of the handle as suggested by their position in the drawings.

Using the fact that for $v \in H$, $g_{v,a_4} = g_{v,a_5}$ the two tours depicted in Figures 8(i) and (ii) yield that $g_{b_3,a_4} = g_{b_3,a_5}$, one proves similarly that $g_{b_1,a_4} = g_{b_1,a_5}$ and $g_{b_2,a_4} = g_{b_2,a_5}$. Using a symmetric set of edges relatively to the handle one proves that $g_{a_i,b_4} = g_{a_i,b_5}$ for $i \in \{1, 2, 3\}$.

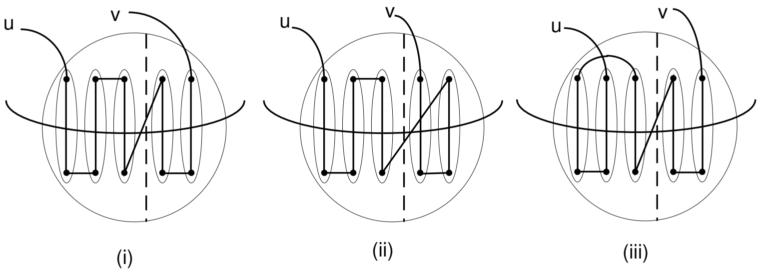

Figure 8: Traces of tight tours

Comparing the set of edges of Figures 8(i) and (iii), yields that $g_{a_1,a_3} = g_{a_2,a_3}$. One can prove in the same way that say $g_{a_1,a_2} = g_{a_2,a_3}$, and therefore the coefficient g_e is the same for all edges of $\gamma(\{a_1, a_2, a_3\})$. Using sets of edges symmetric, relatively to the handle, to those one just used, one gets that the coefficient g_e is the same for all edges of $\gamma(\{b_1, b_2, b_3\})$ (although maybe different from the the value of the previous case). Using the just obtained result and comparing the set of edges of Figure 8(i) and Figure 9(i) one gets that $g_{b_2,a_4} = g_{b_3,a_4}$, and using similar set of edges that $g_{b_1,a_4} = g_{b_2,a_4}$, which yields, with our first result, that the value of g_e is the same for all $e = (b_i, a_j)$, with $i \in \{1,2,3\}$ and $j \in \{4,5\}$. One proves similarly that g_e is the same for all $e = (a_i, b_j)$, with $i \in \{1,2,3\}$ and $j \in \{4,5\}$.

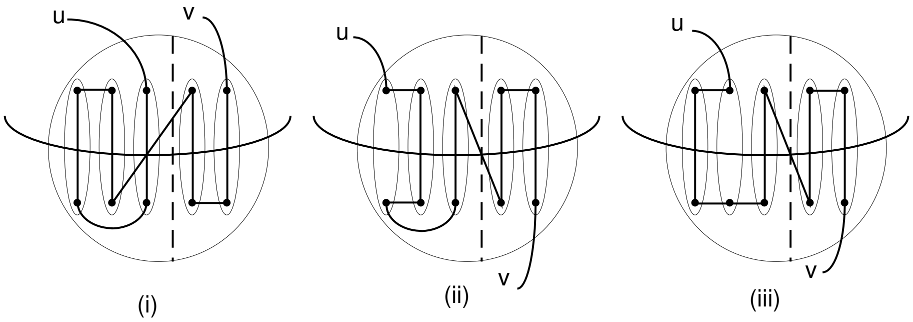

Figure 9: Traces of tight tours

Figures 9(ii) and (iii) yield that $g_{a_1,b_1} = g_{a_2,b_2}$ (we use here the results on $\gamma(\{a_1, a_2, a_3\}$ and $\gamma(\{b_1, b_2, b_3\})$. One can derive similarly that $g_{a_2,b_2} = g_{a_3,b_3}$, and therefore one has $g_{a_1,b_1} = g_{a_2,b_2} = g_{a_3,b_3}$. Doing the same on the other half domino (figure not given), one gets that $g_{a_4,b_4} = g_{a_5,b_5}$ (here we use the result that $g_{a_3,b_4} = g_{a_3,b_5}$).

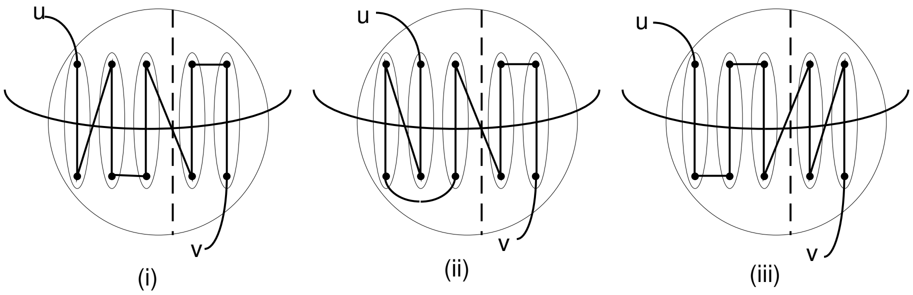

Figure 10: Traces of tight tours

Figures 10(i) and (ii) yield that $g_{a_1,b_2} = g_{b_1,a_2}$ and similar tours yield that $g_{a_i,b_j} = g_{b_i,a_j}$ for $i, j \in \{1,2,3\}$, $i \neq j$. One obtains similarly that $g_{a_4,b_5} = g_{b_5,a_4}$ using the fact that $g_{b_3,a_4} = g_{b_3,a_5}$ (see Figure 10(iii) for one of the set of edges).

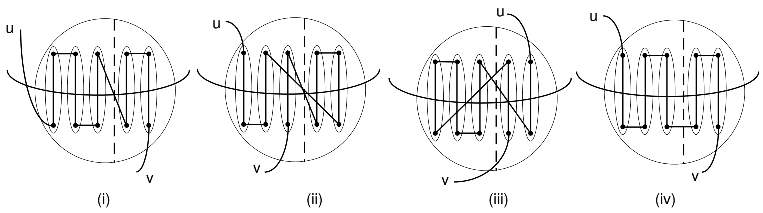

Figure 11: Traces of tight tours

Comparing the set of edges of Figure 10(i) and Figure 11(i), and using the fact that for $u \in H$, one knows that $g_{u,b_1} = d^*_{u,b_1} = d^*_{u,a_1} + 1 = g_{u,a_1} + 1$, one gets $g_{b_1,a_2} = g_{a_1,a_2} + 1$. Taking sets of edges symmetric relative to the handle one gets that $g_{b_2,a_1} = g_{b_1,b_2} + 1$, which yields that $g_{a_1,a_2} = g_{b_1,b_2}$, since one knows that $g_{b_2,a_1} = g_{a_2,b_1}$. Moreover, slightly changing these sets one gets that $g_{b_i,a_j} = g_{a_i,a_j} + 1$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, and therefore all g_{b_i,a_j}, for $i, j \in \{1, 2, 3\}$, $i \neq j$ have the same value. The same can be proved for the other half domino, namely that $g_{a_4,b_5} = g_{a_4,a_5} + 1 = g_{b_4,b_5} + 1 = g_{b_4,a_5}$.

Comparing the set of edges of Figure 10(i) and Figure 11(ii) yields that $g_{a_2,b_5} = g_{a_2,b_1}$, and a symmetric set of edges would give that $g_{b_2,a_5} = g_{b_2,a_1}$. So at this point we know that all g_{a_i,b_j}, $i \neq j$ have the same value except for $g_{a_4,b_5} = g_{a_5,b_4}$. The fact that these two last values equal the preceding ones, is obtained by comparing the set of edges of Figure 10(iii) and Figure 11(iii) and using the just obtained result.

Comparing Figure 9(ii) and Figure 10(ii) one gets the equality $g_{a_1,b_1} + g_{a_2,b_2} = g_{a_1,a_2} + g_{b_1,b_2}$, which, with the fact that $g_{a_2,b_2} = g_{a_1,a_2} + 1$ yields that $g_{a_1,b_1} + 1 = g_{b_1,b_2}$. Letting $g_{a_i,b_i} = \alpha$, to summarize one has:

- $g_{a_i,a_j} = g_{b_i,b_j} = \alpha + 1$ for $i, j \in \{1, 2, 3\}$ or $i, j \in \{4, 5\}$.
- $g_{a_i,b_j} = \alpha + 2$ for $i \neq j$.

Comparing Figure 11(iv) and Figure 10(iii) yields that $g_{b_3,b_4} + g_{a_4,a_5} = g_{b_3,a_4} + g_{b_4,a_5}$, and therefore $g_{b_3,b_4} = \alpha + 3$. One proves similarly that $g_{a_i,a_j} = g_{b_i,b_j} = \alpha + 3$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$.

We are left with determining the value of α. This will be done like in the case of the addition of two teeth. The tours of Figures 12(a) and 12(b) are tight for the domino inequality. The convention is that of the previous proof. There may be more odd teeth, but their number is odd. There may also be more even teeth or none, the result does not change.

One has $g_{u,v} = g_{v,b_3} + g_{u,a_5} - 2 - 2 + 1$, where the two "$-2$" take into account the coboundaries of the two teeth each of the edges (v, b_3) and (u, a_5) intersected in the new set of teeth, the "$+1$", takes into account that the new edge (u, v) crosses the border of the handle. This yields that $g_{b_3,a_5} = 3$ and therefore $\alpha = 1$ and we are done. $\square$

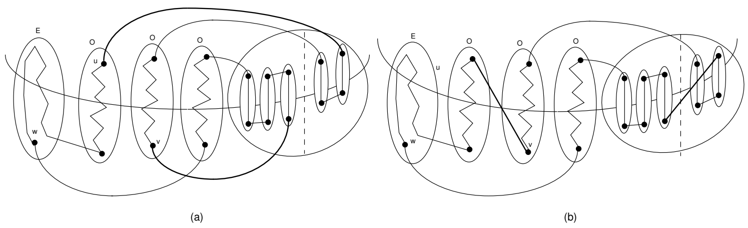

Figure 12: Tight tours for which T_{t+6} is not tight and not tight

6.3 Replacing a minimal tooth by an odd fundamental one

In this section we address the operation of replacing a minimal tooth contained in some half domino or which is a maximal tooth by a fundamental odd tooth. Recall that there are two such teeth, one which contains two minimal teeth per half domino, and another which contains three per half domino. We will refer to them as *(2,2)* and *(3,3)*-odd teeth respectively.

Theorem 14 *Let $\mathcal{D}$ be a facet inducing minimal simple domino configuration. Let T_i be a minimal tooth. Replacing T_i by an odd fundamental tooth yields a facet inducing minimal simple domino configuration $\mathcal{D}^*$.*

Proof. Consider a minimal simple facet inducing domino configuration $\mathcal{D}$, and let $dx \geq d_0$ be the associated domino inequality. Let $T = \{a, b\}$ be one of its minimal teeth. By Theorem 2 one may assume that this minimal tooth is also maximal, that is contained in no other tooth. Let F be a set of $n = |V|$ edges made up of a spanning tree of $V \setminus \{a, b\}$ and three edges (w^*, a), with $w^* \in H$, (a, b) and $(\overline{w}^*, b)$ with $\overline{w}^* \notin H$, such that the unique cycle (to which edge (a, b) belongs) is of odd length. It is easy to convince oneself that it is always possible to find such a set of edges.

Consider the inequality $d'x \geq d'_0$ obtained from the facet inducing domino inequality $dx \geq d_0$ by adding the degree equalities $x(\delta(a)) = 2$ and $x(\delta(b)) = 2$. It is also facet inducing. Let $\mathcal{B}$ be the set of hamiltonian cycles that permit to show that $d'x \geq d'_0$ is facet inducing using F as a set of edges on which the values can be set.

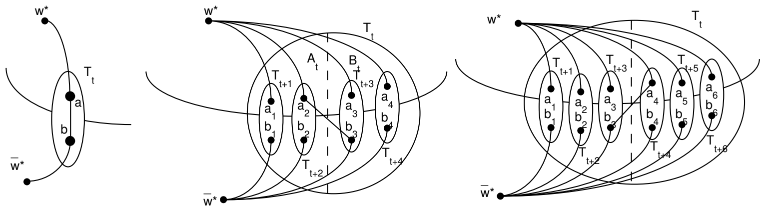

Figure 13: Extensions of the edge set F

Let $\mathcal{D}^*$ be the domino configuration obtained from $\mathcal{D}$ by replacing a minimal maximal tooth, wlog say T_t, by an odd fundamental one T^*. In the figures we assume that the maximal tooth T^* inherits the number of the replaced tooth, that is t. Let $\ell = 4$, if one deals with a $(2,2)$ tooth and $\ell = 6$ else. The nodes of the minimal teeth of that new fundamental tooth are called a_i and b_i, for $i = 1$ to ℓ.

Let w^* and $\overline{w}^*$, be the two nodes defined earlier. Let F^* be the set of edges obtained from F as follows (see Figure 13):

$$F^* = (F \setminus \{(a,b), (w^*,a), (\overline{w}^*,b)\}) \cup_{i=1}^{4} \{(w^*,a_i)\} \cup_{i=1}^{4} \{(\overline{w}^*,b_i)\} \cup \{(a_2,b_3)\}$$

for the $(2,2)$ case and

$$F^* = (F \setminus \{(a,b), (w^*,a), (\overline{w}^*,b)\}) \cup_{i=1}^{6} \{(w^*,a_i)\} \cup \{(\overline{w}^*,b_i)\} \cup \{(b_3,a_4)\}$$

for the $(3,3)$ case. The difference between the two cases is that the edge going from half domino to the other does it slightly differently.

Let $d^*x \geq d_0^*$ be the domino inequality associated to $\mathcal{D}^*$. If it is not facet inducing the face it defines is contained in a facet defined by say $gx \geq g_0$. We set $g_e = d_e^*$ for $e \in F^*$.

We first prove that for all $w \in H$, $g_{w,a_i} = g_{w,a_j}$ for all $i,j \in \{1,\dots,\ell\}$. Let $w \in H$ but not in the same maximal tooth as w^*. Let $\Gamma_{i,j}$ be a tight tour containing edges (w^*,a_i) and (w,a_j), where a_i and a_j are in different half dominoes, tour which exists by Lemma 8 (see Figure 14 with $i = 1$ and $j = \ell$). Let $\Gamma'_{i,j} = \Gamma_{i,j} \setminus \{(w^*,a_i),(w,a_j)\} \cup \{(w^*,a_j),(w,a_i)\}$. Since $g_{w^*,a_i} = d^*_{w^*,a_i} = d^*_{w^*,a_j} = g_{w^*,a_j}$ one has $g_{w,a_i} = g_{w,a_j}$. By changing properly the choices of i and j one proves that, for a given $w \in H$ not in the same maximal tooth as w^*, the value g_{w,a_i} depends only on w and not on i. Letting a node not in the same maximal tooth as w^* play the role of w^* in the previous lines, one proves the same result for all nodes in the handle and in the same maximal tooth as w^*. Therefore for all $w \in H \setminus T^*$, $g_{w,a_i} = \beta_w$ for all i, value which depends only on w.

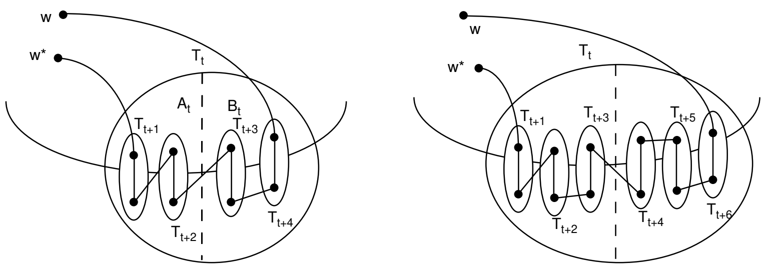

Figure 14: Traces of tight tours

Reversing the role of H and $V \setminus H$ and using $\overline{w}^*$ in the role of w^* one gets the same result for $\overline{w} \notin (H \cup T^*)$, namely, all $g_{\overline{w},b_i} = \overline{\beta}_w$, for all i, value which depends only on $\overline{w}$.

The tours in $\mathcal{B}$, the set of tours used to prove that $d'x \geq d'_0$ is facet inducing, either are tight on the minimal tooth T_t or intersects its coboundary in four edges, (a, w_1), (a, w_2), $(b, \overline{w}_1)$, $(b, \overline{w}_2)$, none of these four edges can belong to $\delta(H)$), since in that case no joker edge is allowed. Therefore the two nodes w_1 and w_2 are in the handle and the two nodes $\overline{w}_1$ and $\overline{w}_2$ are outside the handle. This will have its importance in due time. Let $\Gamma \in \mathcal{B}$ and tight for T_t. We extend Γ to a d^*-tight tour Γ^* as shown in Figure 15. Note that in this case the nodes w and $\overline{w}$ need not be on opposite sides of the handle. Let $\Gamma \in \mathcal{B}$ and not tight for T_t. We extend it to a d^*-tight tour Γ^* as shown in Figure 16. The second drawing of Figure 16 assumes that the two paths obtained from Γ by deleting the four edges incident to T_t have as ends w_1 and $\overline{w}_2$ for the first one, and w_2 and $\overline{w}_1$ for the second one.

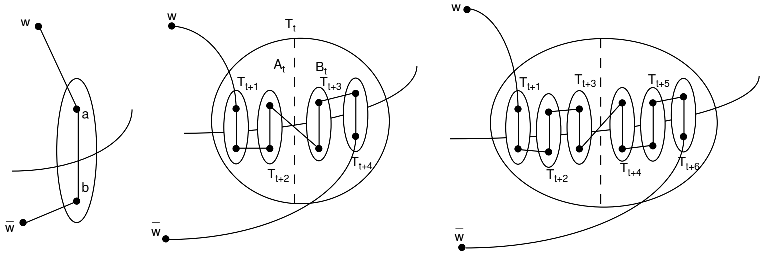

Figure 15: Extension of tours with $x(\delta(T_t)) = 2$

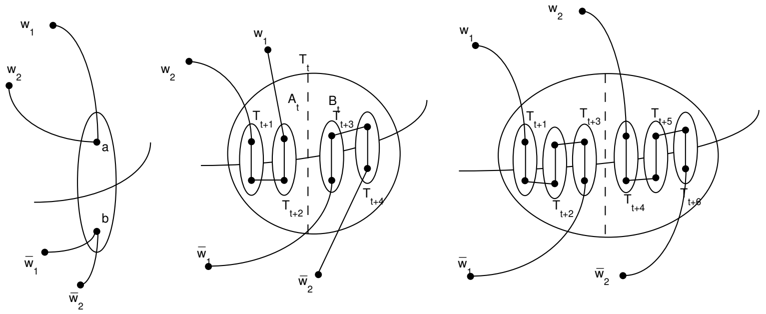

Figure 16: Extension of tours with $x(\delta(T_t)) = 4$

Note that all tours Γ^* for $\Gamma \in \mathcal{B}$, obtained this way, use the same edges in the odd fundamental tooth T_t except for one, the one that crosses from one half domino to the other, but that edge is in F^*, hence one knows its coefficient.

The tours Γ^* for $\Gamma \in \mathcal{B}$ together with F^* therefore enable us to compute all the coefficients g_e for:

- $e \in \gamma(V \setminus T_t)$.
- $e = (w, a_1)$ for all $w \in V \setminus T_t$ (that is even when $w \notin H$).

- $e = (w, b_\ell)$ for all $w \in V \setminus T_t$ (that is even when $w \in H$).

Therefore one knows the coefficients g_e for all $e = (w, a_i)$ for all i and all $w \in H \setminus T_t$ and for all $e = (\overline{w}, b_i)$ for all i and all $\overline{w} \in V \setminus (H \cup T_t)$.

One major difference with the even case, is that one no longer has a symmetry between H and $V \setminus H$. This is because one knows all the coefficients g_{w,a_1} wherever lies w outside T_t, and this is not the case for g_{w,b_1}, which is unknown for $w \in H$. The same holds for g_{w,b_ℓ} which is known wherever lies w outside T_t, which is not the case for g_{w,a_ℓ}.

We now treat the cases $(2, 2)$ and $(3, 3)$ separately.

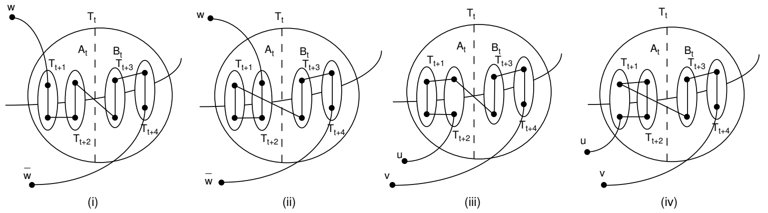

Figure 17: Some traces of tours

Figures 17(i) and (ii) give two set of edges which can be completed to tight tours Γ_1 and Γ_2 by the same set of edges. Comparing these two tours, and using the fact that $g_{w,a_1} = g_{w,a_2}$ one gets that $g_{a_1,b_3} = g_{a_2,b_3} = 3$. One proves similarly that $g_{a_i,b_j} = g_{a_2,b_3} = 3$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

Consider a tight tour for which T_{t+2} is not tight. The trace of such a tour is shown in Figure 17(iii). The tour derived from it by changing only some edges inside T_t is shown in Figure 17(iv). Comparing the two, one gets $g_{a_1,b_1} = g_{a_2,b_2}$ since, as already seen, $g_{a_2,b_3} = g_{a_1,b_3}$. Doing the same on the other half domino one gets $g_{a_3,b_3} = g_{a_4,b_4}$.

Consider a tight tour Γ that contains the edge (a_3, b_4). The trace of such a tour is shown in Figure 18. The second drawing shows a tight tour that differs from Γ by only a few edges. Comparing these tours and using the fact that $g_{u,a_4} = g_{u,a_3}$, one gets that $g_{a_4,b_3} = g_{a_3,b_4}$. Similarly one can prove that $g_{a_1,b_2} = g_{a_2,b_1}$. Now comparing with the set of edges of the last drawing of that same figure, one gets $g_{a_4,b_3} = g_{a_3,b_4} = g_{a_1,b_2} = g_{a_2,b_1} = 3$.

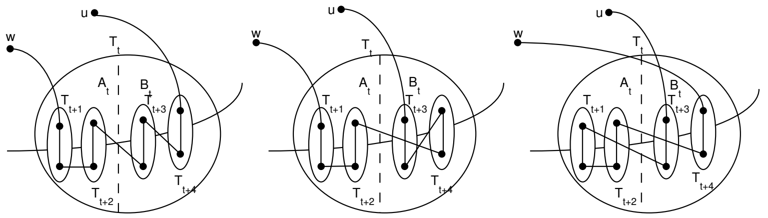

Figure 18: Some traces of tours

The sets of edges of the three drawings of Figure 19 can all be completed by the same set of edges to tight tours. We derive from these sets that $g_{a_4,b_1} = g_{a_4,b_2}$ and similarly, using a tight tour for which T_{t+3} is not tight, that $g_{a_3,b_1} = g_{a_3,b_2}$. Now comparing the set of edges of the third drawing of Figure 19, to the one obtained by replacing the edges (b_2, a_3) and (a_4, v) by the edges (b_2, a_4) and (a_3, v), one gets that $g_{b_2,a_3} = g_{b_2,a_4}$, and therefore the coefficients g_{a_i,b_j}, for $i \in \{3, 4\}$ and $j \in \{1, 2\}$ are all identical. By comparing the set of edges which are the symmetric, relative to the handle of those shown in two last drawings of Figure 18 and using the fact that one already knows that $g_{a_3,b_4} = 3$, one gets that the common value is 3.

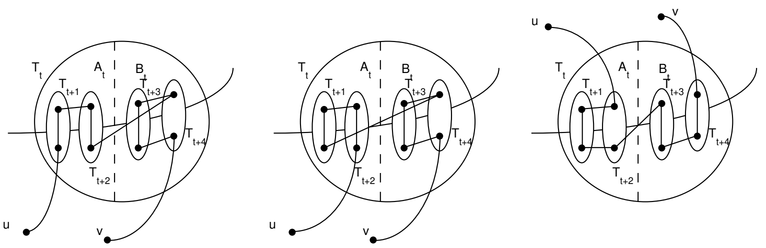

Figure 19: Some traces of tours

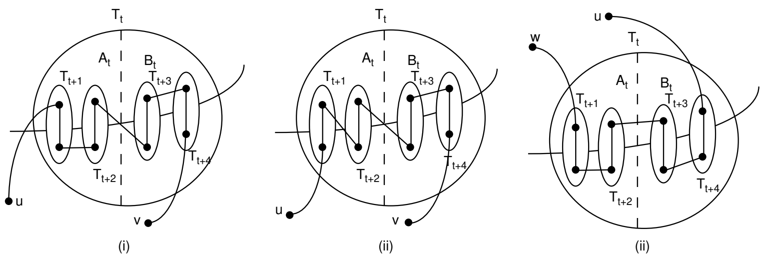

Figure 20: Some traces of tours

Comparing the tours of Figures 20(i) and (ii) and using the fact that we know the value of g_{u,a_1} for $u \notin H \cup T_t$ and since that value is $g_{u,a_1} = g_{u,b_1} + 1$, one gets that $g_{a_1,b_2} = g_{b_1,b_2} + 1$, and therefore $g_{b_1,b_2} = 2$. One can prove in the same way that $g_{a_3,a_4} = 2$. At this point one does not know anything about g_{a_1,a_2} and g_{b_3,b_4}. Now comparing the set of edges of the last drawing of Figure 18 with the one obtained by replacing the five edges (u, a_3), (w, a_4), (b_1, b_2), (a_1, b_3) and (a_2, b_4) by the five edges (u, a_1), (w, a_2), (b_3, b_4), (b_1, a_3) and (b_2, a_4), since the only unknown coefficients are those of the edges (a_1, a_2) and (b_3, b_4), one gets $g_{a_1,a_2} = g_{b_3,b_4}$. Take a tight tour that crosses the tooth T_t as shown in Figure 16, and invert the roles of each half domino. One gets that $g_{a_1,a_2} + g_{b_3,b_4} = g_{a_3,a_4} + g_{b_1,b_2} = 4$, and therefore $g_{a_1,a_2} = g_{b_3,b_4} = 2$.

Comparing now the tour of Figure 17(iii) to that of the first tour of Figure 19, one gets $g_{a_4,b_4} = 1$. Similarly one gets $g_{a_i,b_i} = 1$ for all $i = 1, \dots, 4$.

Comparing the tours of the first drawing of Figure 18 to that of Figure 20(iii), and using the values already calculated, one gets $g_{a_2,a_3} = 4$. One can derive in the same way the coefficients $g_{a_i,a_j} = g_{b_i,b_j} = 4$, for all $i = 1, 2$ and $j = 3, 4$. Now that we know the coefficient of all edges inside tooth T_t it is a triviality to see that $g_{w,a_i} = g_{w,a_1}$ for all w for which we do not already know the value, that is $w \notin (H \cup T_t)$, and that $g_{w,b_i} = g_{w,b_4}$, for all $w \in (H \setminus T_t)$. This ends the proof of the $(2,2)$ case and now turn to the $(3,3)$ case.

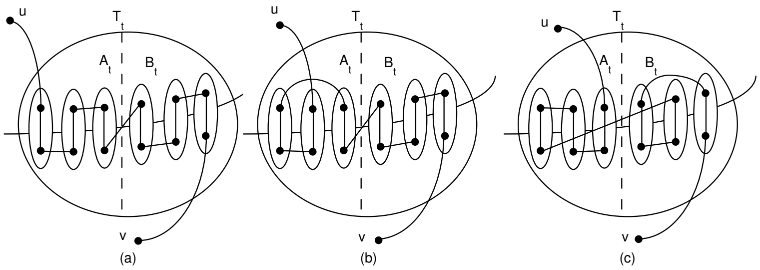

Figure 21: Traces of tours in a $(3,3)$ odd tooth

All the edges of the three sets shown in Figure 21 can be completed by the same set of edges to yield a tight tour. Comparing the edges of the two first sets one gets $g_{a_1,a_3} = g_{a_2,a_3}$. Comparing the edge set of the third set with the one obtained deleting the two edges (a_1, a_2) and (a_3, u) and replacing them by (a_1, a_3) and (a_2, u), one gets that all the edges of $\gamma(\{a_1, a_2, a_3\})$ have the same coefficient, since $g_{a_2,u} = g_{a_3,u}$. One can prove similarly that it is the case for the edges of $\gamma(\{a_4, a_5, a_6\})$, of $\gamma(\{b_1, b_2, b_3\})$ and finally of $\gamma(\{b_4, b_5, b_6\})$, although these coefficient have no reason, at this point, to be equal, from one set to the other.

Using this last result, comparing the sets of the first and last drawing of Figure 21, one can see that $g_{b_1,a_5} = g_{b_3,a_4} = 3$ (recall that (b_3, a_4) is in F^*). Similarly one can prove that $g_{b_i,a_j} = 3$ for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. Using a symmetric set of edges, relatively to the handle, one can prove that all the coefficients g_{a_i,b_j} for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$ are identical, but at this point one cannot derive that common value.

From the two sets of edges of Figure 22, one deduces that $g_{a_1,b_1} = g_{a_2,b_2}$. One similarly proves that $g_{a_1,b_1} = g_{a_3,b_3}$, and therefore $g_{a_1,b_1} = g_{a_2,b_2} = g_{a_3,b_3}$. The proof that $g_{a_4,b_4} = g_{a_5,b_5} = g_{a_6,b_6}$ is identical.

From the two first sets of edges of Figure 23, one deduces that $g_{a_1,b_2} = g_{a_2,b_1}$, and changing slightly the set of edges that $g_{a_i,b_j} = g_{a_j,b_i}$, for $i \neq j$, $i, j \in \{1, 2, 3\}$. Comparing the sets of the first and last drawing of that same figure, one gets $g_{a_2,b_1} = g_{a_3,b_1}$, and the result that all coefficients g_{a_i,b_j} for

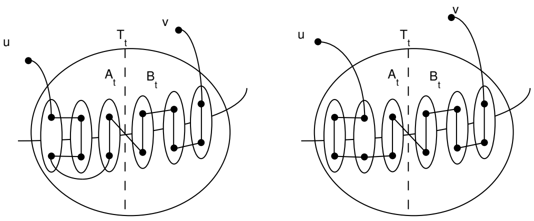

Figure 22: Traces of tours in a $(3,3)$ odd tooth

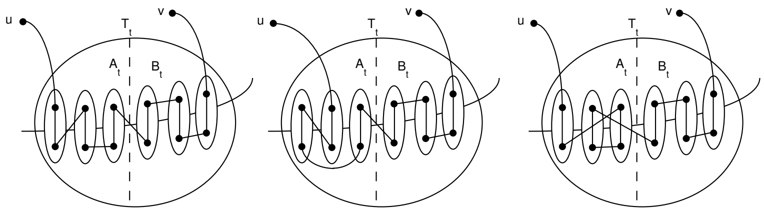

Figure 23: Traces of tours in a $(3,3)$ odd tooth

$i \neq j$, $i,j \in \{1,2,3\}$ are identical follows easily. Similarly one proves the same result for g_{a_i,b_j} for $i \neq j$, $i,j \in \{4,5,6\}$.

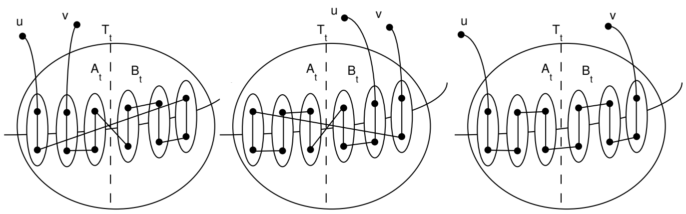

Figure 24: Traces of tours in a $(3,3)$ odd tooth

Comparing the first set of edges of Figure 23 with that of the first set of edges of Figure 24, since $g_{b_1,a_6} = 3$ one gets $g_{b_1,a_2} = 3$, and therefore $g_{a_i,b_j} = 3$, for $i \neq j$, $i,j \in \{1,2,3\}$. Similarly $g_{a_i,b_j} = 3$, for $i \neq j$, $i,j \in \{3,4,5\}$.

Since for $e = (w, a_1)$ with $w \in V \setminus (H \cup T_t)$ one knows that $g_{wa_1} = d'_{wa_1} = d'_{wb_1} + 1 = g_{wb_1} + 1$, comparing the first and second sets of edges of Figure 25, one gets that $g_{a_1,b_2} = g_{b_1,b_2} + 1$, and since we just saw that $g_{a_1,b_2} = 3$ one gets $g_{b_1,b_2} = 2$, and from there that $g_{b_i,b_j} = 2$ for $i,j \in \{1,2,3\}$, $i \neq j$. Similarly one gets $g_{a_i,a_j} = 2$ for $i,j \in \{4,5,6\}$. Unlike in the case of a $(2,2)$ tooth one cannot derive yet the values of g_{b_i,b_j} for $i,j \in \{4,5,6\}$ and g_{a_i,a_j} for $i,j \in \{1,2,3\}$, in both case with $i \neq j$. But inverting the roles of the two half dominoes for the set of edges of the first drawing of Figure 24, one gets the set shown in the second drawing, and by symmetric difference one gets that $g_{a_2,a_3} = g_{a_4,a_5} = 2$. Using sets symmetric relatively to the handle one gets $g_{b_4,b_5} = g_{a_2,a_3} = 2$. Therefore one has $g_{b_i,b_j} = 2$ for $i,j \in \{4,5,6\}$

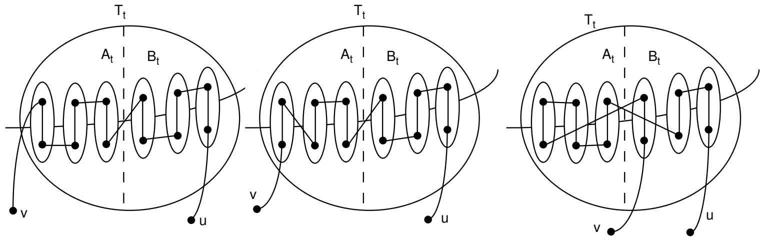

Figure 25: Traces of tours in a $(3,3)$ odd tooth

and $g_{a_i,a_j} = 2$ for $i, j \in \{1,2,3\}$, in both case with $i \neq j$.

Comparing the set of edges of the first drawings of Figures 22 and 24 one gets $g_{a_1,b_1} = 1$ and from there, $g_{a_i,b_i} = 1$ for $i \in \{1,2,3\}$. Similarly, $g_{a_i,b_i} = 1$ for $i \in \{4,5,6\}$.

Comparing the set of edges obtained by taking the symmetric set relative to the handle of the first drawings of Figures 22 and 24, one gets that the $g_{a_i,b_j} = 3$ for $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$.

At this point the only edges with both extremity in T_t for which one does not know yet the coefficient are the penalized edges. Comparing the two sets of edges represented in the two last drawings of 24 shows that $g_{b_3,b_4} = 4$, the other values of g_{b_i,b_j} and g_{a_i,a_j} for $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$ can be derived similarly.

We are left with the edges (w, b_i) for $w \in H \setminus T_t$ and $i \in \{2,3,4,5,6\}$ and (w, a_i) for $w \notin (H \cup T_t)$ and $i \in \{1,2,3,4,5\}$. Comparing the set of edges of the two drawings of Figure 26 and using all the previous computed values one gets $g_{w,b_1} = g_{w,b_6} = d'_{w,b_6} = d'_{w,b_1}$, and in the same way $g_{w,b_i} = g_{w,b_6} = d'_{w,b_6} = d'_{w,b_i}$ for all $i \in \{1,2,3\}$. Letting now the edge (w, b_1) play the (w, b_6), one proves the same result for $i \in \{4,5\}$. The other case is obtained by symmetry relative to the handle. And this ends the proof. $\square$

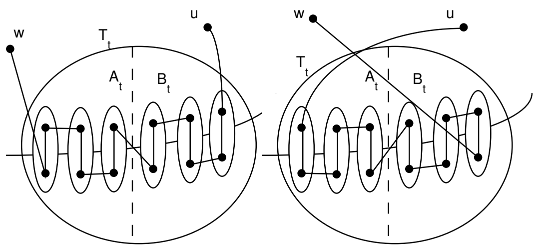

Figure 26: Traces of tours in a $(3,3)$ odd tooth

7 Node Lifting

7.1 General tools for node lifting

In previous sections we studied only simple inequalities. In such inequalities all edge coefficients are non zero. In this section we use tools developed by Naddef and Rinaldi [7] to derive facet inducing results for non simple domino inequalities from results found on simple domino inequalities. The following definitions and theorems will be useful.

Let $K_n = (V_n, E_n)$ and $K_{n^*} = (V_{n^*}, E_{n^*})$ be two complete graphs with n nodes and $n^* > n$ nodes respectively. We denote by $V_n = \{u_1, \ldots, u_n\}$ the nodes of K_n and by $V_{n^*} = V_n \cup \{u_{n+1}, \ldots, u_{n^*}\}$ the nodes of K_{n^*}. We say that an inequality $f^*x^* \geq f_0^*$ defined on $\mathbb{R}^{E_{n^*}}$ is obtained by *node lifting* of the inequality $fx \geq f_0$ defined on $\mathbb{R}^{E_n}$ if

$$f^*_{u_i,u_j} = f_{u_i,u_j} \forall 1 \leq i < j \leq n$$

A special case of node lifting occurs when $f^*_{u,v} = 0$ for some $u \in V_n$ and all $v \in \{u_{n+1}, \ldots, u_{n^*}\}$. We say in this case that the inequality $f^*x^* \geq f_0^*$ is obtained from $fx \geq f_0$ by 0-*node lifting of node* u. A 0-node lifting enables to replace one node in a simple domino configuration by a clique of nodes.

We call 1-*node lifting* a node-lifting that is not a 0-node lifting and satisfy $n^* = n + 1$. A 1-node lifting enables to restore a node which is not in a simple minimal domino configuration.

For any inequality $fx \geq f_0$ and for each node $u \in V$, we define the set $\Delta_f(u)$ by :

$$\Delta_f(u) = \{(v, w) \in E : v \neq u, w \neq u, f_{vw} = f_{uv} + f_{uw}\}.$$

We denote the set of f-tight hamiltonian cycles by $\mathcal{H}_f^=$ and the set of f-tight closed walks by $\mathcal{T}_f^=$. A closed walk Γ is *almost hamiltonian in* u if $u \in V$ has degree 4 in Γ and every other node in V has degree 2 in Γ.

A *basis* for a facet of GTSP(n) defined by an inequality $fx \geq f_0$ is a set of cardinality $|E|$ of f-tight closed walks whose incidence vectors are linearly independent.

A basis $\mathcal{C}_f$ for a facet of GTSP(n) defined by an inequality $fx \geq f_0$ is called *canonical* if it contains $|E| - n$ hamiltonian cycles and n almost hamiltonian walks.

Let e_1 and e_2 be two distinct edges in E. We say that e_1 and e_2 are f-*adjacent* if there exists a tight hamiltonian cycle $\Gamma \in \mathcal{H}_f^=$ containing both e_1 and e_2.

Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two distinct edges in E and let w be a node in V. We say that e_1 and e_2 are f-*adjacent in* w if :

1. e_1 and e_2 belong to $\Delta_f(w)$

2. there exists a closed walk $\Gamma_w \in \mathcal{T}_f^=$ almost hamiltonian in w that contains edges (w, u_1), (w, v_1), (w, u_2) and (w, v_2)

3. $\Gamma_w - (w, u_1) - (w, v_1) + e_1$ and $\Gamma_w - (w, u_2) - (w, v_2) + e_2$ are hamiltonian cycles

The last two conditions can be replaced by "*There exists a $\tilde{f}$-tight hamiltonian cycle containing e_1 and e_2 on the complete graph with node set $V \setminus \{w\}$, where $\tilde{f}$ is induced by f on $\gamma(V \setminus \{w\})$*".

A set of edges $F \subseteq E$ is said to be *f-connected* if for every pair of distinct edges e_1 and $e_2 \in F$ there exists a sequence of k edges $e'_1, \ldots, e'_k$ in F, with $e'_1 \equiv e_1$ and $e'_k \equiv e_2$, such that e'_i is f-adjacent to e'_{i+1}, for $i = 1, \ldots, k-1$.

A set of edges $F \subseteq E$ is said to be *f-connected in w* if for every pair of distinct edges e_1 and $e_2 \in F$ there exists a sequence of k edges $e'_1, \ldots, e'_k$ in E (not necessarily belonging to F), with $e'_1 \equiv e_1$ and $e'_k \equiv e_2$, such that e'_i is f-adjacent in w to e'_{i+1}, for $i = 1, \ldots, k-1$.

The next theorem will be very useful to prove that 1-node lifting of a facet-defining domino inequality is facet-defining.

Theorem 15 (Naddef and Rinaldi [7]) *Let $fx \geq f_0$ be a tight triangular inequality that is facet-defining for STSP(n). An inequality $f^*x^* \geq f_0^*$ which is obtained by 1-node lifting of $fx \geq f_0$ is facet-defining for STSP($n+1$) if it is tight triangular and there exists an edge set $F \subseteq \Delta_{f^*}(u_{n+1})$ and a canonical basis $\mathcal{C}_f$ of $fx \geq f_0$ such that :*

1. *$F \cap \Gamma \neq \emptyset$ for all $\Gamma \in \mathcal{C}_f$*

2. *for all $e \in F$ there exist $e' \neq e, e' \in \Delta_{f^*}(u_{n+1})$ and a f-tight hamiltonian cycle Γ such that e and e' belong to Γ*

3. *every connected component of the graph (V_n, F) contains at least one odd cycle*

4. *F is f^*-connected in u_{n+1}*

The following theorem is useful to prove that a facet-defining domino configuration can be transformed in another facet-defining domino configuration by a zero-lifting.

Theorem 16 (Naddef and Rinaldi [7]) *Let $f^*x^* \geq f_0$ be a tight triangular inequality defined on $\mathbb{R}^{E_{n^*}}$ and obtained by zero-lifting of node $u \in V_n$ from a nontrivial tight triangular inequality that is facet-defining for STSP(n). If the edge set $\delta(u) \subseteq E_n$ is f-connected then $f^*x^* \geq f_0$ is facet-defining of STSP(n^*).*

7.2 1-node lifting

We assume that the 1-node liftings are done before the zero-node liftings. Therefore in the following theorem we prove that a simple domino configuration obtained by 1-node lifting from a facet defining simple domino configuration, is also facet defining.

Theorem 17 *Let $fx \geq f_0$ be a facet defining simple domino inequality for STSP(n) and $f^*x^* \geq f_0^*$ be a simple domino inequality obtained by 1-node lifting of $fx \geq f_0$. Then $f^*x^* \geq f_0^*$ is facet-defining for STSP($n+1$).*

Proof. One may assume that the node u_{n+1} is added in the handle H, since H and $V \setminus H$ play the same role. One may also assume that node u_{n+1} is not contained in any tooth by using repeatedly Theorem 2 to maximal teeth. Note that, although starting with $u_{n+1} \in H$, one may end with this transformation with $u_{n+1} \notin H'$, where H' is the final handle. In this case one can just replace H' by $V \setminus H'$, to obtain a domino configuration that yields the same inequality.

We define the set F of Theorem 15 as being the set of all edges with both extremities in the handle linking nodes in different maximal teeth, more formally :
$F = \{e = (u,v) \in \gamma(H) | u \in T_i, v \in T_j, for\ some\ maximal\ teeth\ T_i\ and\ T_j\ i \neq j\}$
We check whether the conditions of Theorem 15 are satisfied by the set F. Given an edge $e = (u,v) \in F$, the coefficient of this edge in the domino configuration is equal to the number of teeth and handle borders that are crossed by this edge. Since u and v are in the handle, one does not cross the border of H. Since the maximal teeth which u and v belong to are distinct, the number of teeth borders crossed by e is the sum of the number of borders crossed by (u_{n+1}, u) and by (u_{n+1}, v). Therefore $e \in \Delta_{f^*}(u_{n+1})$, and it follows that $F \subset \Delta_{f^*}(u_{n+1})$.

We will prove more than Condition 1 of Theorem 15, not only all closed walks of a basis intersect F, but all tight closed walks. Let Γ be a tight closed walk. The domino configuration contains at least three maximal odd teeth. At most one odd maximal tooth may not be visited properly. Let T_1 and T_2 be two odd maximal teeth that are properly visited by Γ, v_1 and v_2 the respective extremities in H of the paths induced by Γ on these two teeth. Let $(v_i, w_i) \in \delta(T_i) \cap \Gamma$, for $i \in \{1,2\}$, be the edges of Γ exiting tooth T_i and incident to v_i. These two edges may not be different, but then it is the edge $(v_1, v_2) \in T$. At most one of these two edges may belong to $\delta(H)$, since it is then the unique possible joker edge, therefore the other belongs to F, and $\Gamma \cap F \neq \emptyset$.

The first condition of the theorem is therefore satisfied.

Let $e = (u,v) \in F$. Let Γ be a hamiltonian cycle such that e belongs to Γ and all teeth of the domino configuration are visited properly. Then a joker edge e' going from one maximal tooth to another maximal one belongs to Γ. By the definition of the coefficient of such an edge, it is easy to see that $e' \in \Delta_{f^*}(u_{n+1})$, and the second condition of Theorem 15 is satisfied.

By the definition of the set F, it is clear that (V_n, F) is connected. Since there are at least three maximal teeth, it obviously contains an odd cycle.

Let $e = (u, v)$ and $e' = (u', v')$ be two distinct edges of F. In order to prove that the set F is f^*-connected in u_{n+1}, we distinguish the following cases :

1. Nodes u and u' belong to the same maximal tooth in distinct half dominoes, v and v' belong to different maximal teeth distinct from the tooth containing u and u'.

2. Nodes u and u' belong to the same half domino of the maximal tooth T, $v \in T_1$ and $v' \in T_2$, T, T_1 and T_2 pairwise different and all maximal.

3. Nodes u, u', v, v' belong to four distinct maximal teeth.

4. Nodes u and u' belong to a same maximal tooth T, nodes v and v' belong to a same maximal tooth $T' \neq T$.

At first, we consider the case where nodes u and u' belong to the same maximal tooth $T = A \cup B$, $u \in A, u' \in B$ and $v \in T^*$ and $v' \in T^{**}$, with T, T^* and T^{**} pairwise distinct. We are in the conditions of Lemma 9 and therefore there exists a tight hamiltonian cycle containing e and e', which proves that they are f^*-connected in u_{n+1}.

We now consider the situation in which nodes u and u' belong to the same half-domino A of the maximal tooth $T = A \cup B$, v and v' belong to different maximal teeth, $T_1 \neq T$ and $T_2 \neq T$ respectively. If T is also minimal, $H \cap T = \{u\} = \{u'\}$. In this case, it follows from Lemma 10 that there exists a hamiltonian cycle Γ containing e and e'. If T is not a minimal tooth, let $v \in T_1$ and $v' \in T_2$, where T_1 and T_2 are maximal teeth. If T_1 (respectively T_2) is minimal, let w_1 be the node v (resp. w_2 be the node v'). Else let w_1 be a node of $T_1 \cap H$ (resp. $T_2 \cap H$) such that w_1 (resp. w_2) does not belong to the same half-domino as v (resp. v'). By Lemma 10 or Lemma 9, depending on whether $v = w_1$ or not, there exists a tight tour containing edges $e = (u, v)$ and (w_1, w_2). By the same lemmas, there exists a tight tour containing edges (w_1, w_2) and $e' = (u', v')$. Which proves that e and e' are f^*-connected in u_{n+1}.

If nodes u, u', v, v' belong to four different maximal teeth, from the previous cases one gets that the two edges $e = (u, v)$ and (v, v') are f^*-connected in u_{n+1}, and so are the two edges (v, v') and $(u', v') = e'$, which proves that e and e' are f^*-connected in u_{n+1}, since the property is transitive.

To finish, we consider the case where nodes u and u' belong to the same maximal tooth T_1, nodes v and v' belong to the same maximal tooth T_2. There exists at least three maximal teeth in the graph, let T_3 be a third maximal tooth and let w be a node of T_3. We know that the two edges $e = (u, v)$ and (v, w) are f^*-connected in u_{n+1}, and so are the two edges (v, w) and $(u', v') = e'$. Therefore e and e' are f^*-connected in u_{n+1}.

This ends the proof that the obtained lifted inequality is facet inducing, provided the initial one is. □

7.3 0-node lifting

We now turn to the problem of replacing the nodes of a simple configuration by a clique. This can be done by 0-node lifting. In fact 0-node lifting is an operation that can be applied to any node, even one that has been obtained by a previous such operation. For sake of simplicity, and because this is all we need in this paper, we will assume that we replace in one operation a node which is the unique element of a class, by a clique of proper size. This is the object of the following theorem.

Theorem 18 *Let u_n be a node of a facet-defining domino configuration $\mathcal{D}$ of $G = (V_n, E_n)$, such that u_n is the only element of its class. Then the domino configuration $\mathcal{D}^*$ obtained by replacing u_n by a clique of size $k+1$ is facet-defining for STSP$(n+k)$.*

Proof. We denote by $fx \geq f_0$ the domino inequality corresponding to $\mathcal{D}$ and let $f^*x^* \geq f_0^*$ be the one corresponding to corresponding to $\mathcal{D}^*$.

The replacement of u_n by a clique of size $k+1$, consists in adding nodes $u_{n+1}, \ldots, u_{n+k}$ in the domino configuration in exactly the same sets that u_n belongs to. Therefore edge (u_n, u_{n+j}) does not cross any border of sets defining $\mathcal{D}$, so $f^*_{(u_n, u_{n+j})} = 0$, moreover for any node $v \in V_n \setminus \{u_n\}$, the edges (u_n, v) and (u_{n+j}, v) cross exactly the same borders and therefore $f^*_{(u_n,v)} = f^*_{(u_{n+j},v)}$, $1 \leq j \leq k$, and this corresponds exactly to our definition of 0-node lifting.

Using an argument often used, one may assume that node u_n is in the handle. Moreover one may assume that either u_n belongs to a minimal tooth, which can be turned into a maximal one, or belongs to no tooth.

In order to prove that the domino inequality $f^*x^* \geq f_0$ is facet-defining, we use Theorem 16, which states that one only has to show that the set of edges $\delta(u_n)$ is f-connected. Let $e = (u_n, v)$ and $e' = (u_n, v')$ be two distinct edges of $\delta(u_n)$. We distinguish following cases, only the last one deals with the case of u_n in a minimal tooth:

1. Nodes v and v' belong to two distinct maximal teeth and at most one does not belong to H.

2. v and v' are in the handle, u_n is not in any tooth, v and v' belong to the same maximal tooth.

3. $v \in H$, $v' \notin H$, v and v' belong to the same maximal tooth T.

4. $v \notin H$ and $v' \notin H$.

5. Node u_n belongs to a minimal and maximal tooth.

At first, let us assume that $u_n \in H \setminus (\cup T_i)$. We consider the case where nodes $v \in T$ and $v' \in T'$, $T \neq T'$, and both teeth are maximal. Let $\mathcal{D}'$ be the domino configuration made up of the same sets as $\mathcal{D}$ but on the graph $G \setminus \{u_n\}$. Using previously used tight tour building techniques, it is easy to build a tight tour Γ containing edge (v, v'). If one of these nodes is not in the handle, the edge (v, v') is the joker edge. The cycle $\Gamma - (v, v') + (u_n, v') + (u_n, v)$ is f-tight since $(v, v') \in \Delta_f(u_n)$. This shows that there exists a tight tour containing edges e and e'.

Let us assume that nodes v and v' belong to the same maximal tooth T and satisfy $v \in H \cap T, v' \in H \cap T$ (Case 2). There exists a maximal tooth T' different from T. Let $w \in T'$. From the previous case, it follows that $e = (u_n, v)$ and (u_n, w) are f-connected, and so are (u_n, w) and (u_n, v'), and therefore e and e' are f-connected.

If nodes v and v' belong to the same maximal tooth T such that $v \in H \cap T, v' \in T \setminus H$ (Case 3), there exists a maximal tooth T' distinct from T. Let $w \in T' \cap H$. It follows from previous results that $e = (u_n, v)$ and (u_n, w) are f-connected, and so (u_n, w) and $(u_n, v') = e'$. Therefore e and e' are f-connected.

Consider the case in which nodes v and v' are outside the handle : $v, v' \notin H$ (Case 4). Let T be a maximal tooth. Let $w \in T \cap H$. It follows from the previous cases that $e = (u_n, v)$ and (u_n, w) are f-connected, and so are (u_n, w) and (u_n, v'), and therefore e and e' are f-connected.

We are left with the case of $\{u_n\} = T \cap H$ for some minimal tooth T (Case 5). Let $e = (u_n, v)$ and $e' = (u_n, v')$ be two edges that belong to $\delta(u_n)$. If either $e \in \gamma(T)$ and $e' \in \delta(T)$, or $e \in \gamma(H)$ and $e' \in \gamma(H)$, by Lemmas 11 and Lemmas 10 there exists a f-tight tour that contains both e and e'. If $v, v' \notin T \cup H$, let $w \in T \setminus H$. Because we are in conditions of the first subcase of this case, there exists a f-tight tour containing the edges e and (u_n, w), and another one containing (u_n, w) and e'. Therefore e and e' are f-connected. This concludes the proof of this theorem. $\square$

8 The facet inducing theorem and Conclusion

Theorem 19 *The domino inequalities are facet inducing for $STSP(n)$.*

Proof. Since any domino configuration can be built, starting from the minimal simple comb with three teeth, as described in the construction procedure of Section 4, and since all the involved operations are facet preserving, the results follows from the fact that the three tooth comb defines a facet of $STSP(6)$. $\square$

Remark 8.1 *If one uses in the construction of the domino configuration only the addition of two minimal teeth and node lifting. Starting with the three tooth comb on six vertices, one gets all the combs. One can extract from this paper a complete proof that comb inequalities are facet inducing.*

This paper almost characterizes all facet inducing domino parity inequalities. The smallest domino parity inequality with unknown status has 28 nodes and consists of an odd fundamental $(2,2)$-tooth and two even fundamental ones. All domino parity inequalities with unknown status were described in Section 3.

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