Total Dual Integrality of Rothblum’s description of the stable matching polyhedron

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Outline

1. Introduction
   - The stable marriage problem
   - Linear programming formulation

2. Rotations
   - Definitions
   - The structure of rotations

3. Calculating integer optimal solutions of (DR)
   - Preliminaries
   - Construction of the solution

4. Further research
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4 Further research
What is a stable marriage scheme?

Given:

- The set of allowed marriages,
- For each man, a preference order of the women he can marry,
- For each woman, a preference order of the men she can marry.
What is a stable marriage scheme?

Given:
- The set of allowed marriages,
- For each man, a preference order of the women he can marry,
- For each woman, a preference order of the men she can marry.

**Definition**

A **stable marriage scheme** is an allowed matching between men and women with following property:
- There is no man-woman pair who prefer each other to their partners (if they have one).
Formal definition

Bipartite preference system

- $G = (U, V; E)$ a bipartite graph
- For every $w \in U \cup V$, $\prec_w$ a linear order of the edges incident to $w$. 
Formal definition

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Black: $U$, white: $V$. Arrows point to the preceding edge in the order
Formal definition

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Black: \( U \), white: \( V \). Arrows point to the preceding edge in the order

Definition

Edge \( e \) dominates edge \( f \) if \( \exists w \in U \cup V : e \leq_w f \).
Formal definition

$\varphi(e)$: the set of edges which dominate $e$

$\varphi(e)$
Formal definition

$$\varphi(e):$$ the set of edges which dominate $$e$$

Definition

A matching $$M$$ is a **stable matching** if $$\varphi(e) \cap M \neq \emptyset$$ for every edge $$e$$. 
Basic properties of stable matchings

- (Gale, Shapley, 1962) Every bipartite preference system has a stable matching.
Basic properties of stable matchings

- **(Gale, Shapley, 1962)** Every bipartite preference system has a stable matching.
- **(Conway)** The stable matchings of a bipartite preference system have a lattice structure.
Basic properties of stable matchings

- (Gale, Shapley, 1962) Every bipartite preference system has a stable matching.
- (Conway) The stable matchings of a bipartite preference system have a lattice structure.
- There is a $U$-optimal stable matching ($M_U$) and a $V$-optimal stable matching ($M_V$).
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The stable marriage polytope

\( D(w) \): The set of edges of \( E \) incident to \( w \).

**Theorem (Rothblum, 1992)**

The following linear system describes the convex hull of stable matchings:

\[
\begin{align*}
    x & \geq 0, \\
    x(D(w)) & \leq 1 \quad \text{for every } w \in U \cup V, \\
    x(\varphi(e)) & \geq 1 \quad \text{for every } e \in E
\end{align*}
\]

Corollary
A minimum cost stable matching can be found in polynomial time.
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*A minimum cost stable matching can be found in polynomial time.*
The dual linear system

$c$: cost function on the edges

$\psi(e)$: the set of edges which are dominated by $e$

The dual of Rothblum’s system when minimizing $cx$

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\begin{align*}
  y(e) &\geq 0 \quad (e \in E) \quad (1) \\
  \pi(w) &\geq 0 \quad (w \in U \cup V) \quad (2) \\
  y(\psi(uv)) - \pi(u) - \pi(v) &\leq c(uv) \quad \text{if } uv \in E, \quad (3) \\
  \max \sum_{e \in E} y(e) - \sum_{w \in U \cup V} \pi(w). &\quad (4)
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Is there a better way to solve the primal system (PR) and dual system (DR) than general LP techniques?
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Motivation for introducing rotations

- Introduced by Gusfield and Irwing (1987)
- Nice description of the structure of stable matchings
- \((PR)\) can be reduced to a max flow problem on rotations
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Our contribution

Integer optimal solutions to \((DR)\) can be obtained from the dual solution of the max flow problem.

Corollary

*The system \((PR)\) is totally dual integral.*
Definition of rotations

**Definition**

A rotation \( \rho \) is a cycle in \( G \) for which there is a stable matching \( M \) s.t.

1. every second edge of \( \rho \) is in \( M \),
2. the edges ranked between two edges of \( \rho \) are not in any stable matching
3. \( M \Delta \rho \) is a stable matching which covers \( M \) in the lattice
Definition of rotations

**Definition**

A **rotation** $\rho$ is a cycle in $G$ for which there is a stable matching $M$ s.t.

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---

The figure illustrates a rotation $\rho_1$ and a modified stable matching $M_U$. The notation $M \Delta \rho$ refers to the symmetric difference of $M$ and $\rho$.
Definition of rotations

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$\rho_1$

$M_U \Delta \rho_1$
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How rotations determine stable matchings?

Elimination of $\rho$ from $M$: $M \rightarrow M \Delta \rho$

**Fact**

All stable matchings can be obtained by eliminating a sequence of rotations from $M_U$. 

Theorem (Gusfield, Irving)

The rotations of a system can be computed in $O(n^2)$ time.
How rotations determine stable matchings?

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To compute stable matchings using rotations, we have to
1. Compute all rotations
2. Characterize valid elimination sequences
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Elimination obstacles

All rotations in our example:

$\rho_2$ $\rho_1$ $\rho_3$
Elimination obstacles

All rotations in our example:

- $M_U \Delta \rho_2$ is not a matching $\rightarrow \rho_2$ cannot be eliminated from $M_U$
Elimination obstacles

All rotations in our example:

- $M_U \Delta \rho_2$ is not a matching $\rightarrow$ $\rho_2$ cannot be eliminated from $M_U$

- $M_U \Delta \rho_3$ is not stable $\rightarrow$ $\rho_3$ cannot be eliminated from $M_U$
Digraph of elimination obstacles

\textbf{Definition}

Let the directed graph \( D = (R, A) \) contain the following arcs:

- \((\rho, \rho')\) is a \textbf{type 1 arc} if they have an edge in common
- \((\rho, \rho')\) is a \textbf{type 2 arc} if there is an edge \((u, v)\) which is between the two edges of \(\rho'\) at \(u\) and is between the two edges of \(\rho\) at \(v\)
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Digraph of elimination obstacles

\( R \): The set of rotations

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Let the directed graph \( D = (R, A) \) contain the following arcs:

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The digraph \( D \) can be computed in \( O(n^2) \) time.
Properties of $D$

Theorem (Gusfield, Irving)

*There is a one-to-one correspondence between the stable matchings and the 0-indegree sets of $D$.***
Properties of $D$

Theorem (Gusfield, Irving)

There is a one-to-one correspondence between the stable matchings and the 0-indegree sets of $D$.

For an integer cost function $c$ and a rotation $\rho = (v_1, u_1, \ldots v_k, u_k)$ let $c'(\rho) := -c(v_1 u_1) + c(u_1 v_2) - c(v_2 u_2) + c(u_2 v_3) \cdots + c(u_k v_k)$. 
Properties of $D$

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![Diagram of rotations and cost functions](chart.png)
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**Theorem (Gusfield, Irving)**

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For an integer cost function $c$ and a rotation $\rho = (v_1, u_1, \ldots, v_k, u_k)$ let $c'(\rho) := -c(v_1 u_1) + c(u_1 v_2) - c(v_2 u_2) + c(u_2 v_3) \cdots + c(u_k v_k)$.

**Fact**

Then a minimum $c$-cost stable matching corresponds to a minimum $c'$-cost 0-indegree set of $D$. 
Properties of $D$

Polyhedron of the 0-indegree sets of $D$

$$0 \leq x \leq 1,$$

$$x(\rho) - x(\rho') \geq 0 \quad \text{if} \ (\rho, \rho') \in A.$$
Properties of $D$

Polyhedron of the 0-indegree sets of $D$

\[ 0 \leq x \leq 1, \]
\[ x(\rho) - x(\rho') \geq 0 \quad \text{if} \ (\rho, \rho') \in A. \]

- The constraint matrix is totally unimodular
- Optimal integer primal and dual solutions can be obtained using max flow

Corollary

A minimum cost stable matching can be found in $O(n^4)$ time.
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The dual of the 0-indegree system

$$\Delta^+(\rho):$$ set of edges of $D$ leaving $\rho,$
$$\Delta^-(\rho):$$ set of edges of $D$ entering $\rho.$

Integer optimal dual solution

There is a vector $z \in \mathbb{Z}^{RU+A}_+$ such that

$$-z_\rho + z(\Delta^+(\rho)) - z(\Delta^-(\rho)) \leq c'(\rho)$$

for every $\rho \in R,$

$$\sum_{\rho \in R} z_\rho = c(M_U) - c(M_{opt}).$$
The dual of the 0-indegree system

\[ \Delta^+(\rho) : \text{set of edges of } D \text{ leaving } \rho, \]
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**Integer optimal dual solution**

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for every \( \rho \in R \),

\[ \sum_{\rho \in R} z_\rho = c(M_U) - c(M_{opt}). \]
Modified DR

Using the vector $z$, we want to construct a vector $y \in \mathbb{Z}^E$ that satisfies:

\[
\begin{align*}
    y(e) & \geq 0 & \text{if } e \in E \setminus E_{st}, \\
    y(\psi(e)) & \leq c(e) & \text{if } e \in E_{st}, \\
    \sum_{e \in E} y(e) &= c(M_{opt}),
\end{align*}
\]

Where $E_{st}$ is the set of edges that appear in at least one stable matching.

This $y$ can be easily turned into an integer optimal solution of (DR).
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### Modified DR (MDR)

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y(e) & \geq 0 & \text{if } e \in E \setminus E_{st}, \\
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Where $E_{st}$ is the set of edges that appear in at least one stable matching.

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In the following we describe the construction of $y$. 
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Construction of $y$

$\rho_1, \rho_2 \ldots \rho_r$: a topological order of $D$

## Construction method

Recursively define vectors $y_0, y_1, \ldots, y_r$ in $\mathbb{Z}^E$ such that

1. $y_t(e) \geq 0$ if $e \in E \setminus E_{st}$,
2. $y_t(\psi(e)) \leq c(e)$ if $e \in M_U \cup \rho_1 \cup \rho_2 \cup \ldots \cup \rho_t$,
3. $\sum_{e \in E} y_t(e) = c(M_U) - \sum_{i=1}^{t} z_i$.

Here $y_r$ is an integer dual optimal solution of (MDR).
Construction of $y$

$\rho_1, \rho_2 \ldots \rho_r$: a topological order of $D$

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$$y_0(e) := \begin{cases} c(e) & \text{if } e \in M_U, \\ 0 & \text{otherwise} \end{cases}$$
Construction of $y$

$\rho_1, \rho_2 \ldots \rho_r$: a topological order of $D$

Construction method

Recursively define vectors $y_0, y_1, \ldots, y_r$ in $\mathbb{Z}^E$ such that

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Here $y_r$ is an integer dual optimal solution of (MDR).
An almost good vector

Let $\rho_t = (e_0^t, e_1^t, \ldots, e_{2k}^t)$ where $e_0^t$ is better at $U$ than $e_1^t$.

Lemma

*Using only edges of $\rho_t$ and $z_t$, we can define vectors $y_t'$ that satisfy all conditions except those for the edges $e_0^t$.*
An almost good vector

Let $\rho_t = (e^t_0, e^t_1 \ldots e^t_{2k})$ where $e^t_0$ is better at $U$ than $e^t_1$.

**Lemma**

*Using only edges of $\rho_t$ and $z_t$, we can define vectors $y'_t$ that satisfy all conditions except those for the edges $e^t_0$.*

$y'_t$ in our example:

\[ y'_3 : \]

![Diagram showing vectors and edge conditions](image)
Correcting $y'_r$

Correction:

If $(\rho_l, \rho_t)$ is of type 1

If $(\rho_l, \rho_t)$ is of type 2
Correcting $y'_r$

**Correction:**

If $(\rho_l, \rho_t)$ is of type 1

If $(\rho_l, \rho_t)$ is of type 2

In our example:
Theorem

After the corrections the vectors $y_i$ satisfy all conditions.

In our example the integer dual optimal solution is:

$y_3$: 

```
  1  2  1  1  1
-1  1 -1  1 -1
0  -1  1  1  2
1  1  1 -1
```
Is there a known class of TDI systems which includes this?
Further research

- Is there a known class of TDI systems which includes this?
- Generalizations of stable marriage:
  - Stable $b$-matchings
  - Preference systems with partial orders
  - Preference systems with choice functions

Thank you!
Further research

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