THE SYMMETRIC TRAVELING SALESMAN POLYTOPE REVISITED

Denis Naddef
Laboratoire de Informatique et Distribution,
Institut National Polytechnique de Grenoble
e-mail: naddef@imag.fr

Yves Pochet
CORE and IAG, Université Catholique de Louvain
e-mail: pochet@core.ucl.ac.be

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Abstract

We present in this paper a tour of the symmetric traveling salesman polytope, focusing on inequalities that can be defined on sets of nodes. Most of the widely known inequalities are of this type. Many papers have appeared which give increasingly complex valid inequalities for this polytope, but little intuition on why these inequalities are valid has been given.

In order to help in understanding these inequalities, we develop an intuition into their validity by giving a unifying way of defining them through a sequential lifting procedure. This procedure is based on lifting the slack variables associated with subtour elimination inequalities defined on sets of nodes (called teeth). We apply this procedure to some known classes of valid inequalities for the TSP, respectively Comb, Brush, Star, Path, and Bipartition inequalities, where the lifting coefficients are sequence independent. For Comb, Star and Bipartition inequalities, we provide new and non inductive proofs of validity directly inspired by this lifting procedure. We also give an example where a facet defining inequality is derived from the lifting procedure, but where the lifting coefficients are sequence dependent. We finally study the Ladder inequalities and show that they can be generated by an extension of the general procedure, where the lifted variables are different from the slack variables of subtour elimination inequalities.

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1 Introduction

Given a complete undirected loopless graph $G = (V, E)$, to every hamiltonian cycle Γ of G one can assign a vector $x \in \mathbb{R}^E$, where $\mathbb{R}^E$ represents the set of vectors indexed by E, such that $x_e^\Gamma = 1$ if $e \in \Gamma$ and $x_e^\Gamma = 0$ if not. The vector x^Γ is called either the characteristic vector or the representative vector of Γ. The *Traveling Salesman Polytope* (TSP polytope) is the convex hull of the set of all the representative vectors of hamiltonian cycles of G.

This polytope has received a great deal of attention because of its central role in exact solution procedures for the Traveling Salesman Problem. Many papers have appeared which give more and more complex valid inequalities for this polytope (see Chvatal (1973), Grötschel and Padberg (1979a), Grötschel and Pulleyblank (1986), Cornuéjols et al. (1985), Fleischmann (1988), Naddef (1992), Naddef (1990), Naddef and Rinaldi (1991), Naddef and Rinaldi (1992), Boyd and Cunningham (1991), Boyd et al. (1993)).

Unfortunately, except for the so called subtour elimination inequalities, no intuitive idea on why these inequalities are valid has been given. We propose here to give the reader a tour of the Traveling Salesman Polytope, focusing on inequalities that can be defined on sets. Most known inequalities are all of this type. We will give the reader an intuition into the validity of these inequalities. This intuitive approach will not in general be a proof of validity, but, we hope, will greatly help in understanding these inequalities. We will also give a new general way of defining such inequalities.

Section 2 we Section 6 with the Ladder inequalities. procedure that will the Chain inequalities of their validity

The paper is structured as follows. In Section 2 we define the general notation for the inequalities we will study. In Section 3, we present a general procedure for the generation of valid inequalities for the symmetric TSP polytope. This procedure is based on lifting the slack (integer) variables associated with the subtour elimination inequalities defined on some node sets (called teeth). In Sections 4 to 7, we apply this procedure to study some known classes of valid inequalities for the TSP, respectively Combs in Section 4, Brushes in Section 5, Star and Path inequalities in Section 6, and Bipartition inequalities in Section 7. It is shown that all these inequalities can be generated by the lifting procedure, and that in these cases lifting is sequence independent. In Section 8, we study the Ladder Inequalities and show that they can be generated by an extension of the sequence independent lifting procedure. In Section 9, we show an example of a facet defining inequality generated by the lifting procedure, and for which lifting is sequence dependent. We conclude in Section 10 with a generalization of the Chain Inequalities of Padberg and Hong (1980) for which an intuitive explanation of their validity remains to be given.

We end this section with some notation: Let $S \subset V$. Then $\gamma(S)$ represents the set of edges with both endnodes in S and $\delta(S)$ the set of edges with exactly one endnode in S, i.e. $\gamma(S) = \{(u,v) \in E : u, v \in S\}$ and $\delta(S) = \{(u,v) \in E : u \in S, v \notin S\}$. The edge set $\delta(S)$ is in general called the *coboundary* of S (some authors say *cocycle* of S). We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$. For $S \subset V$ and $T \subset V \setminus S$ we denote by $(S : T) = (T : S)$ the set of edges with one endnode in S and the other in T. For $E^\star \subset E$ we let $x(E^\star)$ represent $\sum_{e \in E^\star} x_e$. Let G(S) denote the induced subgraph on S. i.e. $G(S) = (S, \gamma(S))$.

2 CLOSED SET FORM INEQUALITIES

Let $\mathcal{S} = \{S_1, ..., S_i, ..., S_p\}$ be a set of distinct (not necessarily non intersecting) subsets of V. Let $\alpha_1, \alpha_2, ..., \alpha_i, ..., \alpha_p$ be strictly positive integers. An inequality is said to be in *closed set form* if it can be written as:

$$\sum_{i=1}^{p} \alpha_i x(\delta(S_i)) \geq r(\mathcal{S}) \tag{2.1}$$

or equivalently as:

$$\sum_{i=1}^{p} \alpha_i x(\gamma(S_i)) \leq \sum_{i=1}^{p} \alpha_i \mid S_i \mid - r(\mathcal{S})/2 \tag{2.2}$$

where $r(\mathcal{S})$ is the same in both inequalities and can be seen as a "rank function", depending on some structural property of $\mathcal{S}$. The equivalence between 2.1 and 2.2 can easily be seen using the following Relation 2.3, valid for any subset S of V and obtained by summing up all the equations $x(\delta(v)) = 2$ for $v \in S$ (edges inside S are counted twice, those of the coboundary once):

$$2x(\gamma(S)) + x(\delta(S)) = 2 \mid S \mid \tag{2.3}$$

Historically, the Form 2.2 of the inequalities has been the first one used. This is its only advantage. The authors strongly advocate, for reasons that we give below, the use of Form 2.1. In this paper all inequalities will be given in this form. A first reason is that the right hand side (from now on, abbreviated to RHS) only depends on the structure of $\mathcal{S}$, and not on the number of vertices it contains. A second reason is that it is preserved by taking complements of sets. A third reason is that proofs are in general much easier in that form. Finally, and this is not independent from the previous reason, because we know facts about cycles and coboundaries: a cycle always intersects a coboundary in an even number of edges.

In a sense all inequalities valid inequalities for the Traveling Salesman Polytope have a closed set, if one considers all the two node sets defined by all edges with non zero coefficient. There is a standard form for inequalities for the Traveling Salesman Polytope, that is the tight triangular form, which is unique up to multiplication of both sides by a strictly positive number (see Naddef and Rinaldi (1993)). Not all valid inequalities for the Traveling Salesman Polytope have a tight triangular closed set form. The best example are the hypohamiltonian inequalities (see Cornuéjols et al. (1985)).

One first mystery we will try to explain is the reason why, in all known inequalities in closed set form where the set $\mathcal{S}$ contains more than a single element, the set $\mathcal{S}$ partitions into 2 sets, a set $\mathcal{H} = \{H_1, \ldots, H_i, \ldots, H_h\}$ of handles and a set $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$ of teeth.

The most trivial inequality in closed set form is the one where $\mathcal{S}$ contains a unique element, that is made of a unique proper subset S of nodes. The inequality is known as the subtour elimination inequality $x(\delta(S)) \geq 2$. Its validity is trivial since any hamiltonian cycle must use, because of connectivity, at least one edge of the coboundary of $S \neq V$, and by parity it must use at least 2. A set $S \subset V$ is said to be *tight* for a hamiltonian cycle Γ with characteristic vector x if $|\Gamma \cap \delta(S)| = 2$, or equivalently $x(\delta(S)) = 2$.

Remark 2.1 *Everything that will be said in this paper also holds for spanning closed walks, and therefore can be used for the Graphical Traveling Salesman Polytope, the most used, today, relaxation of the Traveling Salesman Polytope (see Cornuéjols et al. (1985), Fleischmann (1988), Naddef (1992), Naddef (1990), Naddef and Rinaldi (1991) and Naddef and Rinaldi (1992)).*

3 A UNIFYING FRAMEWORK

We consider first the general case of closed set form valid inequalities for the TSP, and define a general procedure to derive valid inequalities. Applications of this general framework will be given in the subsequent sections.

For a closed set form inequality, let the set $\mathcal{S}$ of subsets of V be partitioned into two sets $\mathcal{H}$ and $\mathcal{T}$ where $\mathcal{H} = \{H_1, \ldots, H_i, \ldots, H_h\}$ is the set of so called *handles*, $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$ is the set of so called *teeth*. Let $\alpha_1, \ldots, \alpha_i, \ldots, \alpha_h$ be given non-zero integers associated with the handles.

The inequalities in closed set form that are known and that we are trying to explain in this paper are all written as follows:

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) + \sum_{j=1}^{t} \beta_j x(\delta(T_j)) \geq A + 2 \sum_{j=1}^{t} \beta_j \tag{3.1}$$

or equivalently:

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) \ \geq \ A \ - \ \sum_{j=1}^{t} \beta_j \ [x(\delta(T_j)) - 2]. \tag{3.2}$$

Expression 3.2 suggests the following interpretation, which can be used in order to define a general procedure for computing the coefficients A and β_j, $j = 1, \cdots, t$, of the valid inequality.

When all teeth are tight for a given hamiltonian cycle Γ $(x^\Gamma(\delta(T_j)) = 2$ for all $j = 1, \cdots, t)$, the inequality 3.2 reduces to $\sum_{i=1}^h \alpha_i x(\delta(H_i)) \geq A$. Hence, for validity, the largest possible value of A is the minimum value of the left hand side of 3.2 over all hamiltonian cycles Γ with all teeth tight. Formally,

$$A \;=\; \min_{\Gamma} \{\sum_{i=1}^{h} \alpha_i \mid \Gamma \cap \delta(H_i) \mid : \;\mid \Gamma \cap \delta(T_j) \mid = 2 \text{ for all } j = 1, \cdots, t\} \quad (3.3)$$

Now we describe a general procedure to compute the coefficients β_j in the valid inequality 3.2. First we order the teeth. We assume without loss of generality that the teeth have been renumbered in such a way that their order is $T_1, \cdots, T_j, \cdots, T_t$. Next, we compute the coefficients β_j from $j = 1$ to $j = t$ using a sequential lifting procedure. The lifted variables are the nonnegative integer slack variables s_j associated to the subtour elimination inequalities defined on the teeth, namely $s_j = x(\delta(T_j)) - 2$ for $j = 1, \cdots, t$. Initially, to compute the coefficient A in expression 3.3, all slacks s_j are projected to zero and we obtain the inequality $\sum_{i=1}^h \alpha_i x(\delta(H_i) \geq A$ which is valid for the subspace where $s_j = 0$ for $j = 1, \cdots, t$. Then, sequentially from $j = 1$ to $j = t$, all slacks s_j are lifted in.

For $k \in \{1, \cdots, t\}$, assuming coefficients $\beta_1, \cdots, \beta_{k-1}$ have been already computed, the lifting coefficient β_k is taken as the minimum value β for which the inequality

$$\sum_{i=1}^{h} \alpha_i x^\Gamma(\delta(H_i)) + \sum_{j=1}^{k-1} \beta_j [x^\Gamma(\delta(T_j)) - 2] + \beta[x^\Gamma(\delta(T_k)) - 2] \geq A \qquad (3.4)$$

is satisfied by all hamiltonian cycles Γ with $s_j = x^\Gamma(\delta(T_j)) - 2 = 0$ for all $j > k$. If we let

$$b_k^l \;= \min_{\Gamma \in \mathcal{G}} \;\left[\sum_{i=1}^{h} \alpha_i x^\Gamma(\delta(H_i)) + \sum_{j<k} \beta_j [x^\Gamma(\delta(T_j)) - 2]\right] \qquad (3.5)$$

where the minimum is taken over all hamiltonian cycles $\Gamma \in \mathcal{G} = \{\Gamma : x^\Gamma(\delta(T_j)) = 2, \text{ for all } j > k, \; x^\Gamma(\delta(T_k)) = 2l\}$, and where $b_k^l = \infty$ if $\mathcal{G} = \emptyset$, then the value of β_k is defined by

$$\beta_k = \max_{l>1}(b_k^1 - b_k^l)/(2l-2) \qquad (3.6)$$

Note that by construction of β_k, we must have $b_k^1 = A$ for all $k \in \{1, \cdots, t\}$. Also note that the β_k's are not in general integers. The previous construction is summarized in the following Theorem.

Theorem 3.1 *Let $\mathcal{H} = \{H_1, \ldots, H_i, \ldots, H_h\}$, let $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$, be an ordered set of teeth, and let $\alpha_1, \ldots, \alpha_i, \ldots, \alpha_h$ be non-zero integers associated with the handles. If A is defined by Expression 3.3, and, for $k \in \{1, \cdots, t\}$, β_k is the coefficient of the tooth T_k determined by Expressions 3.5 and 3.6, then the following inequality is valid:*

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) \geq A - \sum_{j=1}^{t} \beta_j \left[x(\delta(T_j)) - 2\right] \tag{3.7}$$

Proof: Trivial by the way we constructed the coefficients of the teeth. If a hamiltonian cycle does not satisfy the inequality, let $T_{j^\star}$ be the last non tight tooth in the order. Then the coefficient $\beta_{j^\star}$ has not been computed correctly. $\diamondsuit$

Remark 3.1 *The above procedure provides a way of defining valid inequalities for the TSP. The difficulty in using Theorem 3.1 to prove validity of inequalities is the exponential growth in the number of cases to study when the number of teeth increases. Nevertheless, in most cases, and in all known inequalities in closed set form, the lifting procedure can be observed to be sequence independent. This allows one to develop an intuitive explanation of the validity of these inequalities.*

In other words, to better understand these valid inequalities, and to provide an intuitive non algebraic combinatorial validity argument, it suffices to understand how to compute A and, for each tooth T_j, β_j as if T_j was the first in the lifting order. Roughly writing, A is the (weighted) minimum number of times a tour "crosses the borders" of the handles when all teeth are tight. Hence, the teeth play the role of forcing sets. β_j can be interpreted as the "best gain" (on this minimum number of crossings of the borders of the handles) per unit of increase of the number of crossings of the border of T_j.

Such combinatorial arguments will be given in the coming sections for the known inequalities in closed set form, which can all be defined as applications of Theorem 3.1. For a given inequality, the lifting interpretation also indicates which cycles satisfying it at equality should be looked for in order to prove that it is facet inducing.

Remark 3.2 *This intuitive approach does not provide in general a proof of validity. Proving, for some class of inequalities, that the coefficients can be computed irrelevant of the order, that is independently, does not seem easy.*

However, for the Comb, Star, Path, and Bipartition inequalities, we provide direct, non algebraic and non inductive proofs of validity. All these *proofs follow the same structure inspired by the closed set form inequality in Expression 3.7, and by its lifting interpretation.*

Remark 3.3 *For some ordering of the teeth, it can happen that the coefficient β_j of a tooth T_j is zero in Expression 3.6. This only means that the forcing set T_j has no influence on the left hand side of the inequality.*

Remark 3.4 *We call degenerate a tooth for which the maximum in Expression 3.6 is not given by $l = 2$. This definition of degeneracy, as we will see in Section 7, is more restrictive than that given by in Boyd and Cunningham (1991) for the Bipartition inequalities.*

Remark 3.5 *If, in the Binested inequalities, we change the coefficients of the degenerate teeth by the stronger coefficients obtained by Expression 3.6, (in this case lifting is sequence independent too, and one may assume the tooth considered to be the first one in Expression 3.5) then we get a family of inequalities which contains all known inequalities in closed set form, that is including bipartition inequalities. This family of inequalities can be obtained by the above lifting procedure, with sequence independent lifting coefficients. Note that the Ladder inequalities studied in Section 8 are not in closed set form.*

Remark 3.6 *Finally, some known inequalities that are not in closed set form can be similarly interpreted by a modified lifting procedure giving inequalities of the form*

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) \geq A - \sum_{j=1}^{t} \beta_j \left[x(\delta(T_j)) - 2\right] \quad - \epsilon(\mathcal{S}) \tag{3.8}$$

where the additional term $\epsilon(\mathcal{S})$ represents a lifting term with a lifted variable different from the slack of a subtour elimination constraint. Ladder inequalities studied in Section 8 correspond to this type of extension.

4 COMB INEQUALITIES

The first non trivial class of inequalities was discovered by Chvatal (1973) and later generalized by Grötschel and Padberg (1979b). These are the Comb inequalities.

Let $\mathcal{H} = \{H\}, \mathcal{T} = \{T_1, T_2, \ldots, T_j, \ldots, T_t\}, t \geq 3$ and odd satisfy:

$$H \cap T_i \neq \emptyset \quad \text{for } i = 1, ..., t \tag{4.1}$$

$$T_i \setminus H \neq \emptyset \quad \text{for } i = 1, ..., t \tag{4.2}$$

$$T_i \cap T_j = \emptyset \quad \text{for } 1 \leq i < j \leq t \tag{4.3}$$

We will say that the set $\mathcal{S} = \{H, \{T_i : i = 1, ..., t\}\}$ satisfying these conditions defines a comb with handle H and teeth T_i, i=1,....,t. If $| V | \geq 6$, the corresponding Comb inequality

$$x(\delta(H)) + \sum_{j=1}^{t} x(\delta(T_j)) \geq 3t + 1 \tag{4.4}$$

which can be rewritten as

$$x(\delta(H)) \geq t + 1 - \sum_{j=1}^{t} \left[x(\delta(T_j)) - 2\right] \tag{4.5}$$

has been shown to be facet inducing for the TSP polytope in Grötschel and Padberg (1979b). Note that Condition 4.3 is in some sense not satisfactory since it is not closed under taking complements. One way of getting around this problem is to require the given sets to satisfy the conditions after eventually replacing some of them by their complements. Figure 1 gives two sets $\mathcal{S}$ that give the same comb inequality, the first is a set as described in 4.1 to 4.3, the other with T_3 replaced by its complement. This section deals with combs described by 4.1 to 4.3. For combs as in the second case of Figure 1, validity can be proved as a particular case of Brush inequalities, which is the topic of the next section. This figure enables us to define our convention for figures that will hold throughout this paper. Sets with a black point have to be nonempty, those with a white filled point may or may not have nodes in them, those with no points must be empty. A point does not represent a unique node but a set of nodes in that position.

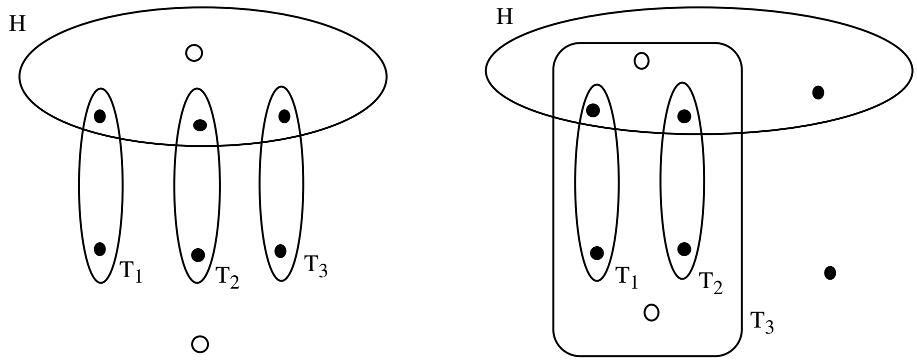

Figure 1: Example of a comb $(t = 3)$

The classical way to prove validity of such an inequality is algebraic. Since no algebraic proof of validity for this inequality directly in the form of 4.5 has been published, we do not resist in giving one here. The reader can compare with the corresponding proof for the inequality written in Form 2.2, and will easily agree that this one is much easier.

Theorem 4.1 *Comb inequalities are valid for the Traveling Salesman Polytope.*

Proof: It is easy to check that we have:

$$\begin{aligned} x(\delta(H)) + \sum_{j=1}^{t} x(\delta(T_j)) &\geq \\ 1/2\left[\sum_{j=1}^{t} x(\delta(H \cap T_j)) + \sum_{j=1}^{t} x(\delta(T_j \setminus H)) + \sum_{j=1}^{t} x(\delta(T_j))\right] &\geq 3t \quad (4.6) \end{aligned}$$

Now the left part of 4.6, $x(\delta(H)) + \sum_{j=1}^{t} x(\delta(T_j))$, is an even integer as sum of only even integers, when x represents a hamiltonian cycle. Since 3t is odd, 3t can be replaced by 3t + 1. $\diamondsuit$

Note that this also proves that if all subtour inequalities are satisfied Inequality 4.5 cannot be violated by more than 1 by x (note that in the traditional Form 2.2, this amount of maximum violation is 0.5).

Unfortunately this easy proof of validity does not give us any insight into what Comb inequalities are saying about hamiltonian cycles (or spanning closed walks). We try now to understand this better using their Expression 4.5 by providing non algebraic proofs of validity. The first proof uses a simple combinatorial inductive argument. We present a second non inductive proof because it is directly related to the general lifting procedure given in Theorem 3.1, and because it can be generalized to similar validity proofs for Star, Hyperstar, Path and Bipartition inequalities.

A non algebraic and inductive proof of validity of Comb inequalities: Assume first that $t = 3$. Take a hamiltonian cycle Γ, if $|\Gamma \cap \delta(T_j)| = 2$ for $j = 1, 2$ and 3, then $|\Gamma \cap \delta(H)| \geq 3$. Since $|\Gamma \cap \delta(H)|$ must be even we have $|\Gamma \cap \delta(H)| \geq 4$ and Inequality 4.5 is satisfied by all x representing such a cycle. If $|\Gamma \cap \delta(T_{j^\star})| \neq 2$, for some $j^\star$, $|\Gamma \cap \delta(T_{j^\star})| \geq 4$, the 3 other sets having at least 2 edges of their coboundaries in Γ, again Inequality 4.5 is satisfied by all x representing such a cycle. Proving validity for the general case can be done by induction on the number of teeth $t \geq 5$. Assume validity for combs with $t-2$ teeth. Take a hamiltonian cycle Γ. If $|\Gamma \cap \delta(T_j)| = 2$ for j=1, ..., t, then $|\Gamma \cap \delta(H)| \geq t$ and by parity one must have $|\Gamma \cap \delta(H)| \geq t+1$ and the result follows for those cycles. Else assume $|\Gamma \cap \delta(T_{j^\star})| \geq 4$, for some $j^\star$, let us assume for simplicity $j^\star = t$. By induction hypothesis we have that: $|\Gamma \cap \delta(H)| + \sum_{j=1}^{t-2} |\Gamma \cap \delta(T_j)| \geq 3t - 6 + 1$ adding $|\Gamma \cap \delta(T_t)| \geq 4$ and $|\Gamma \cap \delta(T_{t-1})| \geq 2$ we prove again that the inequality is valid. $\diamondsuit$

A non algebraic and non inductive proof of validity of Comb inequalities: Let Γ be any hamiltonian cycle, and x its associated characteristic vector.

If all teeth are tight, then there are at least t edges of the coboundary of H in Γ because $x(\delta(T_j)) = 2$ implies $x(T_j \cap H : T_j \setminus H) > 0$. As t is odd and $x(\delta(H))$ even, we must then have $x(\delta(H)) \geq t+1$. So, $A = t+1$ in the lifting procedure defined in Theorem 3.1, and the inequality 4.6 is satisfied by x if all teeth are tight. It is also trivially satisfied whenever $x(\delta(H)) \geq t+1$.

In the other cases, i.e. $x(\delta(H)) < t+1$, define η by $x(\delta(H)) = t+1-2\eta$. Note that $\eta > 0$ is an integer because t is odd, and $x(\delta(H))$ is even. As $(t+1)/2$ represents the minimum number of times a hamiltonian cycle enters H when all teeth are tight, η represents the reduction in the number of times a cycle enters H with respect to the tight teeth case. We prove basically, and the Comb inequality tells us essentially, that the number of non tight teeth has to be at least as large as η

First, we consider each tooth T_j separately, and define $\theta_j = 1$ if $x(T_j \cap H : T_j \setminus H) = 0$ and $\theta_j = 0$ otherwise. It is readily seen that $x(\delta(T_j))/2 \geq 1 + \theta_j$ because $x(\delta(T_j))/2$ is the number of times the tour Γ enters T_j, or equivalently the number of paths in Γ traversing T_j, and the tooth T_j is traversed by at least two paths (i.e. is not tight) when $\theta_j = 1$. Thus, $x(\delta(T_j)) - 2 \geq 2\theta_j$.

Next, we consider the handle for which it is easy to check that $x(\delta(H)) \geq \sum_{j=1}^{t} x(T_j \cap H : T_j \setminus H) \geq \sum_{j=1}^{t}(1 - \theta_j) = t - \sum_{j=1}^{t} \theta_j$. Together with $x(\delta(H)) = t + 1 - 2\eta$, this gives $\sum_{j=1}^{t} \theta_j \geq 2\eta - 1 \geq \eta$ where the last inequality holds because η is a positive integer.

Hence, we have shown that the reduction η in the number of paths traversing H is at most $\sum_{j=1}^{t} \theta_j$, which is at most, by definition of θ_j, the number of non tight teeth. This suffices to prove validity because $x(\delta(H)) = t + 1 - 2\eta \geq t + 1 - 2\sum_{j=1}^{t} \theta_j \geq t + 1 - \sum_{j=1}^{t}[x(\delta(T_j)) - 2]$. $\diamondsuit$

Analyzing the previous proof of validity we understand better the role played by the teeth. When you impose a hamiltonian cycle to be tight on the teeth (enter and exit exactly once), because of parity, you obtain that this cycle has one edge more in the coboundary of the handle than the minimum t required by the tightness of the teeth. Moreover, if only one tooth T_j is split into two parts (enter and exit twice in such a way that $x(T_j \cap H : T_j \setminus H) = 0$), the number of edges of the cycle in the coboundary of the handle is at least $t - 1$, and splitting more than one tooth makes this inequality even easier to satisfy.

Another proof can be found in Applegate et al. (1998) as a particular case of the clique tree inequalities. The spirit of that proof is very similar to our last one.

We will find that this type of argument is a constant in all the inequalities in closed set form that we know of. Teeth force edges of the coboundaries of the handles to be present in the cycle. The parity requirement on coboundaries is not always the reason of that forcing, as we will see right now with the Brushes.

5 2-BRUSHES

2-Brushes have been introduced by Naddef and Rinaldi (1991). They are obtained by a 2-sum composition operation on Combs, which also yields the well known Clique Tree inequalities which we will look at later. Brushes are defined by $\mathcal{H} = \{H\}$, $\mathcal{T} = \{T_1, \ldots, T_t, \ldots, T_{t+k}\}$, which satisfy the following conditions:

$$
\begin{aligned}
&H \cap T_i \neq \emptyset && \text{for } i = 1, ..., t+k && (5.1)\\
&T_i \setminus H \neq \emptyset && \text{for } i = 1, ..., t+k && (5.2)\\
&T_i \cap T_j = \emptyset && \text{for } 1 \leq i < j \leq t && \\
& && \text{or } t+1 \leq i < j \leq t+k && (5.3)\\
&T_i \cap T_j = \emptyset \text{ or } T_i \subset T_j && \text{for } 1 \leq i \leq t \text{ and } t+1 \leq j \leq t+k && (5.4)\\
&\forall i, 1 \leq i \leq t, \exists j(i), t+1 \leq j(i) \leq t+k \text{ such that } T_i \subset T_{j(i)} && && (5.5)\\
&\mid \{i : i \leq t \text{ and } T_i \subset T_j\} \mid \neq 0 \text{ and even for } t+1 \leq j \leq t+k && && (5.6)\\
&H \setminus \textstyle\bigcup_{j=1}^{t+k} T_j \neq \emptyset && && (5.7)\\
&V \setminus (H \cup (\textstyle\bigcup_{j=1}^{t+k} T_j)) \neq \emptyset && && (5.8)
\end{aligned}
$$

In other words there are t "small" teeth and k "big" teeth. Each big tooth contains an even number of small ones. There must be at least a node in the handle (in position "a" of Figure 2) and one outside the handle (in position "b" of Figure 2) which are in no tooth. There is no parity requirement on k but 5.6 implies t even. Figure 2 gives an example of a 2-Brush with 2 big teeth. A comb can be seen as a 2-Brush with a single big tooth.

The 2-Brush inequality is:

$$x(\delta(H)) + \sum_{j=1}^{t+k} x(\delta(T_j)) \quad \geq \quad 3t + 2k + 2 \tag{5.9}$$

which can be rewritten as:

$$x(\delta(H)) \quad \geq \quad t+2 \; - \sum_{j=1}^{t+k} \left[x(\delta(T_j)) - 2 \right] \tag{5.10}$$

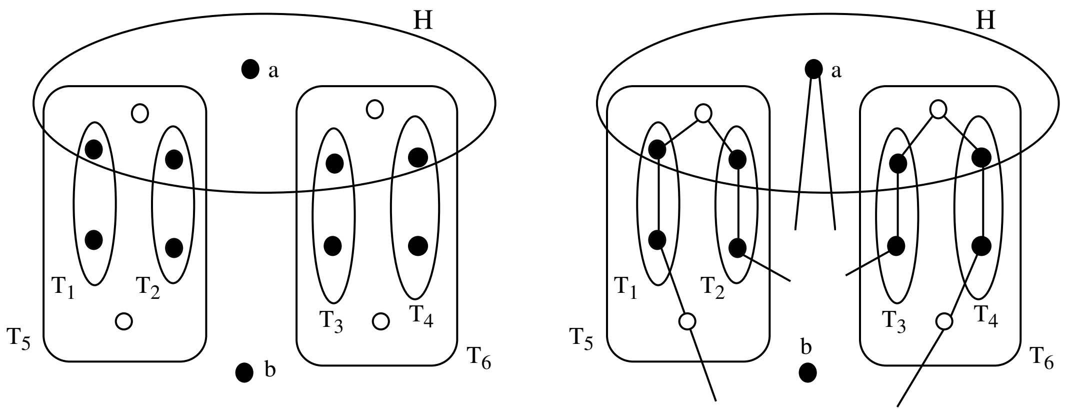

Figure 2: Example a Brush $(t = 4,\ k = 2)$

Why do we need Conditions 5.7 and 5.8 for 5.10 to be valid? The following is not a proof of validity of 5.10 but an explanation on why we need Conditions 5.7 and 5.8.

As shown in Figure 2, the minimum number of edges in the coboundary of H that enables all teeth (small or big) to be tight is t (=number of small teeth). But, if all the teeth are tight for a hamiltonian cycle, there is no way it can pick up all vertices in H and outside H, which are in no tooth, if it is going to use only t edges of the coboundary of H. The most it can do is pick up all the vertices in H or out of H but not both. Since t is even, we need at least 2 more edges in the coboundary of H in that case. Hence, in the lifting procedure defined in Theorem 3.1, we have $A = t + 2$ when both conditions 5.7 and 5.8 are imposed.

Moreover, by splitting any tooth into two parts (traversing it with two separate paths), one reduces by no more than two the number of edges of the cycle in the coboundary of the handle.

6 STAR, PATH and BINESTED INEQUALITIES

Path inequalities (in the class of which we consider also what have been called Wheelbarrow and Bicycle inequalities) have been defined by Cornuéjols et al. (1985), Star inequalities by Fleischmann (1988) and the binested inequalities by Naddef (1992). The Star inequalities properly contain the Path inequalities. The Binested inequalities contain the Star inequalities, but their description here would be too long, so the point we want to make will be done on the Star inequalities. Let $\mathcal{H} = \{H_1, \ldots, H_i, \ldots, H_h\}$, $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$, with t odd, $\alpha_1, \ldots, \alpha_i, \ldots, \alpha_h$ non-zero integers associated with the handles, and $\beta_1, \ldots, \beta_j, \ldots, \beta_t$ integers associated with the teeth such that:

$$H_1 \subset H_2 \subset \cdots \subset H_i, \cdots \subset H_h \tag{6.1}$$

$$H_1 \cap T_j \neq \emptyset \qquad \text{for } j = 1, \ldots, t \tag{6.2}$$

$$T_j \setminus H_h \neq \emptyset \qquad \text{for } j = 1, \ldots, t \tag{6.3}$$

$$T_i \cap T_j = \emptyset \qquad \text{for } 1 \leq i < j \leq t \tag{6.4}$$

$$(H_{i+1} \setminus H_i) \setminus \bigcup_{j=1}^{t} T_j = \emptyset \qquad \forall i, 1 \leq i \leq h-1 \tag{6.5}$$

and the condition that relates the β_j's to the α_i's, which we call the *Interval Property*, holds. To define this property, we need a few definitions. Given a tooth T_j, we define an *interval* relative to T_j to be a maximal index set of (successive) handles which have the same intersection with T_j. Let $I = \{\ell, \ell+1, ..., \ell+r\}$ be an interval, we call *weight* of the interval $\sum_{i=\ell}^{\ell+r} \alpha_i$

The Interval Property: for each tooth T_j, we have $\beta_j \geq$ the maximum weight of an interval relative to T_j

Figure 3 shows a tooth and the traces of the different handles on it. In the tooth, where a node is shown there is at least one node, where none is shown there are no nodes. The handles, as suggested by the drawing, are numbered from top to bottom. There are 4 intervals, $\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}$, of weight 3, 4, 5 and 4, and therefore the β coefficient of that tooth must be at least 5.

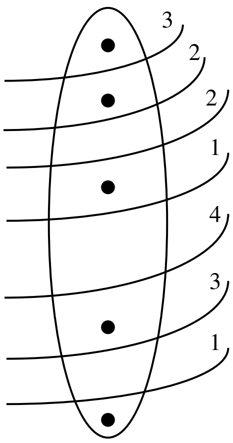

Figure 3: Example of intervals

The Path, bicycle and wheelbarrow inequalities of Cornuéjols et al. (1985) are the special case where all the intervals relative to a same tooth have the same weight and the coefficient of that tooth is exactly this value. Figure 4 gives an example of Path Inequality. The coefficients associated to the handles and teeth are beside each set.

The Path, bicycle and wheelbarrow inequalities are facet inducing for the symmetric TSP polytope. Some star inequalities which are not in that subfamily also are, but no sufficient conditions are known. There are many necessary conditions, all of them are related to forbidding the star inequality to be the sum of known facet inducing inequalities. For example the interval property has to be satisfied to equality, else the star inequality is the sum of another star inequality and some subtour elimination inequalities. The coefficients of the handle and teeth must be indecomposable in the sense that they are not the sums of the coefficients of two or more other star inequalities.

Conjecture 6.1 *All star inequalities that satisfy the interval property to equality and the coefficients of which are indecomposable in the previously defined sense are facet inducing for the symmetric TSP polytope.*

Figure 5 gives a facet inducing star inequality which is not in the subfamily of path inequalities.

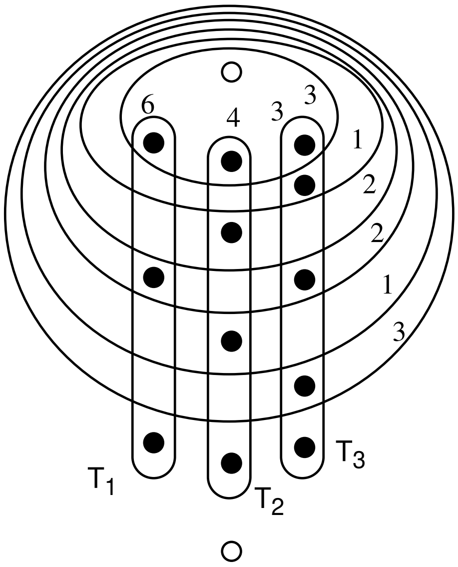

Figure 4: Example of path $(h = 6, t = 3)$

The rationale behind the Interval property is far from being obvious. We will try to explain it, but first let us give the Star inequality:

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) + \sum_{j=1}^{t} \beta_j x(\delta(T_j)) \geq (t+1) \sum_{i=1}^{h} \alpha_i + 2 \sum_{j=1}^{t} \beta_j \tag{6.6}$$

which can be rewritten as:

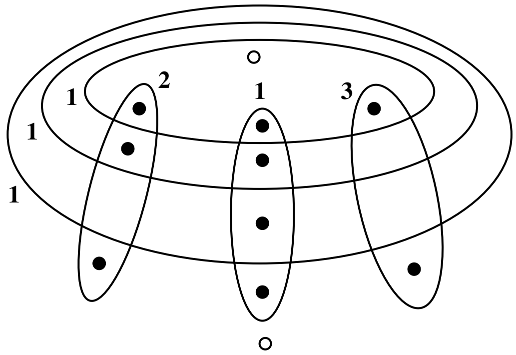

Figure 5: Example of a facet inducing star inequality

$$\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) \geq (t+1) \sum_{i=1}^{h} \alpha_i \; - \; \sum_{j=1}^{t} \beta_j \left[x(\delta(T_j)) - 2 \right] \tag{6.7}$$

The first term in the RHS of 6.7 corresponds to A in the lifting Theorem 3.1 and can easily be understood if you consider a Hamiltonian cycle for which all teeth are tight. Each handle must then have at least $t+1$ edges of that cycle in its coboundary because t is odd (see Comb inequalities), which leads to the first part of the RHS. But so far this is true whatever the α's or β's are.

Consider a Hamiltonian cycle Γ such that $T_{j^\star}$ is the only non tight tooth and $\mid \Gamma \cap \delta(T_{j^\star}) \mid = 4$. Let $\{r, \cdots, s-1\}$ be an interval of $T_{j^\star}$. Assume Γ uses edges of the coboundaries of the handles H_r to H_{s-1} only inside the teeth T_i for $i \neq j^\star$. In other words, $x(T_{j^\star} \cap H_i : T_{j^\star} \setminus H_i) = 0$ for all $i \in \{r, \cdots, s-1\}$ (see Figure 6 where $j^\star = 2$ from Figure 4). Then the coboundaries of the handles H_r to H_{s-1} may contain only t-1 edges, the coboundaries of the other handles at least t+1. Therefore a loss to the LHS, corresponding to the former case, of at most $2\sum_{i=r}^{s-1} \alpha_i$ compensated for by a loss to the RHS of $2\beta_{j^\star}$. Therefore to have validity of the inequality one must have $\beta_{j^\star} \geq \sum_{i=r}^{s-1} \alpha_i$. This argument is valid for all intervals of $T_{j^\star}$ and therefore $\beta_{j^\star}$ must be greater or equal to the largest weight of an interval.

This is not a proof of validity, which we will now give, since we have only considered two types of hamiltonian cycles. This only tries to explain what may otherwise be obscure conditions on the coefficients. It also gives a fast way of computing the RHS of those inequalities. What we have done here for the star inequalities, together with what has been said for the 2-Brushes, carries over word by word to the binested inequalities: the RHS computation, an explanation of the conditions on the coefficients on handles and teeth and the presence of nodes in key places.

A new proof of validity of Star Inequalities: Let Γ be any Hamiltonian

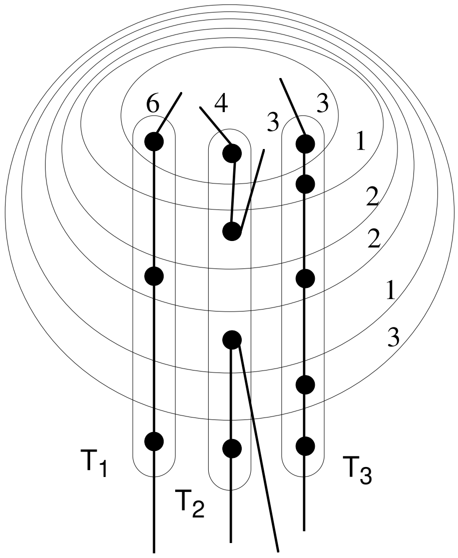

Figure 6: Example of minimal trace with T_2 not tight $(r = 3,\ s - 1 = 4)$

cycle, and x its associated characteristic vector. As in the proof of the validity of the Comb inequality, define η_i by $x(\delta(H_i)) = t + 1 - 2\eta_i$, for $i = 1, \cdots, h$. Again, η_i is integer because t is odd, and $x(\delta(H_i))$ is even, for $i = 1, \cdots, h$.

As before, if all teeth are tight, we must have $x(\delta(H_i)) \geq t + 1$ for each handle H_i, and the Star inequality (6.7) is satisfied by x. It is also trivially satisfied whenever $x(\delta(H_i)) \geq t + 1$ for each $i = 1, \cdots, h$.

We prove now validity in the other cases, i.e. $\eta_i > 0$ for some $i \in \{1, \cdots, h\}$, occurring only if some tooth is not tight. We introduce first some notation. For each handle H_i, $i \in \{1, \cdots, h\}$, and each tooth T_j, $j \in \{1, \cdots, t\}$, define $\theta_{ij} = 1$ if $x(T_j \cap H_i : T_j \setminus H_i) = 0$, and $\theta_{ij} = 0$ otherwise. So, $\theta_{ij} = 1$ indicates that the two sets of nodes $T_j \cap H_i$ and $T_j \setminus H_i$ are separated (no edge from one set to the other) in the cycle Γ.

(i) First, we consider each tooth T_j separately, for $j = 1, \cdots, t$. It is easy to see that

$$x(\delta(T_j)) \geq 2 + 2\left(\frac{\sum_{i=1}^{h} \theta_{ij}\ \alpha_i}{\beta_j}\right)$$

because

- if $\theta_{rj} = 1$ for some $r \in \{1, \cdots, h\}$, then $\theta_{ij} = 1$ for all handles $i \in \{1, \cdots, h\}$ belonging to the same interval of T_j as r,
- if $I_j = \{r, \cdots, s-1\}$ is an interval of T_j with $\theta_{ij} = 1$ for all $i \in I_j$, then $\sum_{i \in I_j} \theta_{ij}\ \alpha_i \leq \beta_j$ by the interval property defining β_j,

- if ℓ is the number of intervals I of T_j with $\theta_{ij} = 1$ for all $i \in I$, then $x(\delta(T_j)) \geq 2(\ell + 1) = 2 + 2\ell$ because the tooth T_j is separated into $\ell + 1$ sets such that there is no edge from one set to another in the cycle Γ, and therefore there are at least two edges in the coboundary of T_j for each of these $\ell + 1$ sets,
- and finally $\sum_{i=1}^{h} \theta_{ij}\ \alpha_i \leq \ell\ \beta_j$.

Hence we have shown that the number of times the cycle Γ enters the set of nodes T_j is at least $1 + \left(\frac{\sum_{i=1}^{h} \theta_{ij}\ \alpha_i}{\beta_j}\right)$.

(ii) Next, we consider each handle H_i separately, for $i = 1, \cdots, h$. It is easy to see that

$$2\eta_i - 1 \leq \sum_{j=1}^{t} \theta_{ij}$$

because $t+1-2\eta_i = x(\delta(H_i)) \geq \sum_{j=1}^{t} x(T_j \cap H_i : T_j \setminus H_i) \geq \sum_{j=1}^{t}(1 - \theta_{ij}) = t - \sum_{j=1}^{t} \theta_{ij}$.

This is a simple argument showing that if there are $t+1-2\eta_i$ edges on the coboundary of H_i in the cycle Γ, then there are at least $(2\eta_i - 1)$ teeth for which $T_j \cap H_i$ and $T_j \setminus H_i$ are separated in the cycle Γ.

(iii) Finally, we put together the observations made on teeth and handles. Using $\eta_i \leq 2\eta_i - 1$ when $\eta_i > 0$, and therefore $\eta_i \leq \sum_{j=1}^{t} \theta_{ij}$ for all $i = 1, \cdots, h$, we obtain

$$\begin{aligned}
\sum_{i=1}^{h} \alpha_i x(\delta(H_i)) &= \sum_{i=1}^{h} \alpha_i(t+1) - \sum_{i=1}^{h} 2\alpha_i\eta_i \\
&\geq \sum_{i=1}^{h} \alpha_i(t+1) - \sum_{i=1}^{h} 2\alpha_i(\sum_{j=1}^{t} \theta_{ij}) \\
&= \sum_{i=1}^{h} \alpha_i(t+1) - \sum_{j=1}^{t} [\sum_{i=1}^{h} 2\theta_{ij}\alpha_i] \\
&\geq \sum_{i=1}^{h} \alpha_i(t+1) - \sum_{j=1}^{t} \beta_j [x(\delta(T_j)) - 2]
\end{aligned}$$

$\diamondsuit$

7 BIPARTITION INEQUALITIES

Bipartition inequalities were introduced by Boyd and Cunningham (1991) in order to generalize the Clique Tree inequalities of Grötschel and Pulleyblank (1986).

These inequalities are defined by a set $\mathcal{H} = \{H_1, \ldots, H_i, \ldots, H_h\}$ of disjoint handles, a set $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$ of disjoint teeth and non-zero rational coefficients $\beta_1, \ldots, \beta_j, \ldots, \beta_t$ associated to the teeth such that:

$$H_i \cap H_j = \emptyset \qquad \text{for } 1 \leq i < j \leq h \tag{7.1}$$

$$T_i \cap T_j = \emptyset \qquad \text{for } 1 \leq i < j \leq t \tag{7.2}$$

$$T_j \setminus H_i \neq \emptyset \qquad \text{for } 1 \leq i \leq h \text{ and } 1 \leq j \leq t \tag{7.3}$$

$$t_j = \mid \{i : T_j \cap H_i \neq \emptyset\} \mid \geq 1 \qquad \text{for } 1 \leq j \leq t \tag{7.4}$$

$$h_i = \mid \{j : H_i \cap T_j \neq \emptyset\} \mid \geq 3 \text{ and odd} \qquad \text{for } 1 \leq i \leq h \tag{7.5}$$

$$\beta_j = 1 \text{ if } T_j \setminus (\textstyle\bigcup_{i=1}^{h} H_i) \neq \emptyset \text{ else } \beta_j = t_j/(t_j - 1) \qquad \text{for } 1 \leq j \leq t \tag{7.6}$$

Therefore no tooth is contained in a handle, each tooth intersects $t_j \geq 1$ handles, and each handle intersects an odd number h_i of teeth.

Teeth T_j such that $T_j \setminus (\bigcup_{i=1}^{h} H_i) = \emptyset$ are called by Boyd and Cunningham *degenerate*. This definition of degeneracy does not match exactly the concept of degeneracy that we developed in Section 3. With our concept, teeth such that $t_j = 2$ and $T_j \setminus (\bigcup_{i=1}^{2} H_i) = \emptyset$ (therefore with $\beta_j = 2$) would not be called degenerate.

The corresponding Bipartition inequality is then:

$$\sum_{i=1}^{h} x(\delta(H_i)) + \sum_{j=1}^{t} \beta_j x(\delta(T_j)) \geq \sum_{i=1}^{h} (h_i + 1) + 2 \sum_{j=1}^{t} \beta_j \tag{7.7}$$

which can be rewritten as:

$$\sum_{i=1}^{h} x(\delta(H_i)) \geq \sum_{i=1}^{h} (h_i + 1) \; - \; \sum_{j=1}^{t} \beta_j \left[x(\delta(T_j)) - 2 \right] \tag{7.8}$$

This inequality falls in our general framework developed in Section 3 for closed set form inequalities. The problem here is to understand the strange coefficients $\beta_j = t_j/(t_j - 1)$.

For $t_j = 2$ this gives a coefficient $\beta_j = 2$ whose interpretation is exactly identical to that given in the Star inequalities (confirming that this case is not particular).

For $t_j > 2$, the coefficient $\beta_j = t_j/(t_j - 1)$ is fractional. We will illustrate its explanation by considering the clique tree inequality of Figure 7a and the very similar bipartition inequality of Figure 7b. They only differ by the fact that in the clique tree there is a node in position "a" in tooth $T^\star$ and none in the case of the shown bipartition inequality. All teeth coefficients not shown are 1.

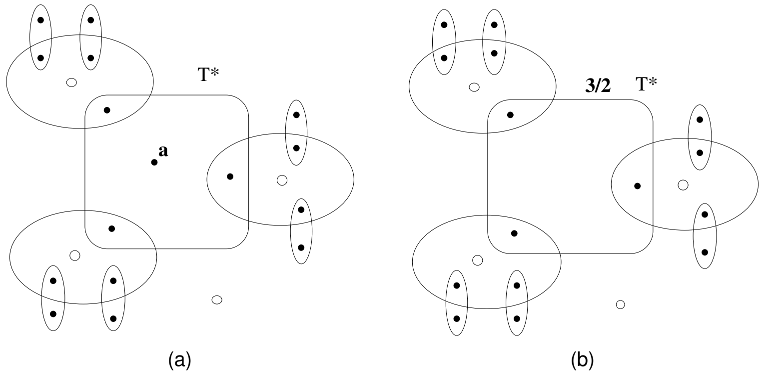

Figure 7: A clique tree and a bipartition $(h = 3,\, t = 7)$

For the bipartition inequality of Figure 7b, the trace of a hamiltonian cycle Γ, such that every tooth is tight and entering the handles a minimum number of times, can be seen in Figure 8a. This gives a cycle for which the inequality is satisfied with equality (tight). A hamiltonian cycle such that all teeth are tight satisfies with equality the bipartition inequality if all the handles intersect it in the minimum number of edges (i.e. one more than the number of teeth it intersects). This is mainly the "spirit" of all these "handle-teeth" inequalities as we tried to show throughout this paper.

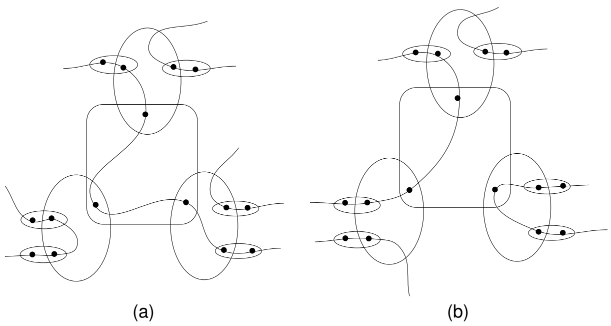

Figure 8: Traces of hamiltonian cycles

Figure 8b gives a trace when the coboundary of tooth $T^\star$ intersects a hamiltonian cycle Γ' in exactly 4 edges, while all other teeth are tight for Γ'. Using the notation of Section 3, this corresponds to lifting tooth $T^\star$ first, and to take $l = 2$ in Expression 3.5. Note that we have in total 2 edges more, than with Γ, on the set of all coboundaries of teeth which compensates for the at most 2 edges less on the set of coboundaries of the handles. Figure

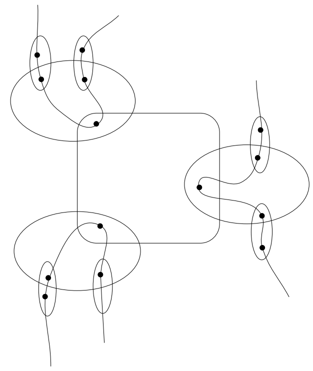

Figure 9: Trace of hamiltonian cycle yielding a “jump”

9 gives a trace when the coboundary of tooth $T^\star$ intersects a hamiltonian cycle Γ” in exactly 6 edges, while all other teeth are tight for Γ”. This corresponds again to lifting tooth $T^\star$ first, but with $l = 3$ in Expression 3.5. Here suddenly, something strange happens: we have in total 2 edges more, than with Γ’, on the set of all coboundaries of teeth which are more than compensated for, by 4 edges less on the set of coboundaries of the handles. This cannot happen in the case of the clique tree of Figure 7a since one must pick up the nodes in position “a”. In clique trees, an augmentation of 2 on the coboundaries of the teeth cannot yield more than a “saving”of 2 on the coboundaries of the handles. We just saw that this is not the case for bipartition inequalities. In fact if tooth $T^\star$, without any node outside any handle, intersects t handles, when we enable a hamiltonian cycle to have $2t$ edges in the coboundary of $T^\star$, that is $2t - 2$ more than the minimum, we can “save” $2t$ edges on the coboundaries of the handles it intersects. Let β be the coefficient of tooth $T^\star$. Therefore, from Expression 3.6 for $l < t$ we obtain $\beta \geq (2l-2)/(2l-2) = 1$, and for $l = t$ we get $\beta \geq 2t/(2t-2)$. This leads to $\beta = t/(t-1)$, the coefficient defined in Condition 7.6.

Note that this is the first case where the coefficient of a tooth is not given by only considering the difference between what happens when a cycle intersects the coboundary in 2 and 4 edges.

Again this is not a proof of validity of the bipartition inequalities, but only an explanation on why Condition 7.6 is needed. We now give a complete proof of validity of Bipartition inequalities. If the reader understood, in the preceding lines, the role of the presence or not of a node in $T_j \setminus \cup_i H_i$ for teeth intersecting three or more handles, the proof will resemble that of the validity of star inequalities. In turn this proof is in spirit very similar to that given in Applegate et al. (1998) for the clique tree inequalities.

Proof of validity of bipartition inequalities:

Γ, x, η_i for $i = 1, \cdots, h$ and θ_{ij} for each Handle H_i intersecting tooth T_j are as in the proof of validity of Star inequalities. In particular $\theta_{ij} = 1$

means that the two sets of nodes $T_j \cap H_i \neq \emptyset$ and $T_j \setminus H_i \neq \emptyset$ are separated (no edge from one set to the other) in the cycle Γ. Therefore for each i such that $\theta_{ij} = 1$, the tour Γ contains at least two edges of the coboundary of T_j which are also in the coboundary of $T_j \cap H_i \neq \emptyset$. Therefore $x(\delta(T_j)) \geq 2\sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} \theta_{ij}$. This lower bound on the number of edges of Γ in the coboundary of T_j is tight only if $T_j \setminus (\cup_{i=1}^h H_i) = \emptyset$ and $\theta_{ij} = 1$ for all i such that $T_j \cap H_i \neq \emptyset$. In all the other cases we can add 2 to that bound, since Γ has other nodes to visit in T_j.

We now have all we need to prove validity.

If all teeth are tight, we must have $x(\delta(H_i)) \geq h_i + 1$ for each handle H_i, and the Bipartition inequality (7.8) is satisfied by x. It is also trivially satisfied whenever $x(\delta(H_i)) \geq h_i + 1$ for each $i = 1, \cdots, h$.

We prove now validity in the other cases, i.e. $\eta_i > 0$ for some $i \in \{1, \cdots, h\}$, occurring only if some tooth is not tight.

(i) For each tooth T_j, $j = 1, \cdots, t$, with $T_j \setminus (\cup_{i=1}^h H_i) \neq \emptyset$, using the above observations and $\beta_j = 1$ in this case, we have

$$x(\delta(T_j)) \geq 2 + 2 \sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} \theta_{ij} = 2 + 2\frac{1}{\beta_j} \sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} \theta_{ij}$$

(ii) For each tooth T_j, $j = 1, \cdots, t$, with $T_j \setminus (\cup_{i=1}^h H_i) = \emptyset$, we have

$$x(\delta(T_j)) \geq 2+2\frac{(t_j - 1)}{t_j} \sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} \theta_{ij} = 2+2\frac{1}{\beta_j} \sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} \theta_{ij}$$

Note this bound is only tight when either all $\theta_{ij} = 0$ or all $\theta_{ij} = 1$.

(iii) Next, we consider each handle H_i separately, for $i = 1, \cdots, h$. It can be shown easily that

$$\eta_i \leq \sum_{j=1,\cdots,t:T_j\cap H_i\neq\emptyset} \theta_{ij}$$

(iv) Finally, putting together the observations made on teeth and handles, we obtain

$$
\begin{aligned}
\sum_{i=1}^{h} x(\delta(H_i)) &= \sum_{i=1}^{h}(h_i+1) - \sum_{i=1}^{h} 2\eta_i \\
&\geq \sum_{i=1}^{h}(h_i+1) - \sum_{i=1}^{h}\Big(\sum_{j=1,\cdots,t:T_j\cap H_i\neq\emptyset} 2\theta_{ij}\Big) \\
&= \sum_{i=1}^{h}(h_i+1) \;-\; \sum_{j=1}^{t}\Big[\sum_{i=1,\cdots,h:H_i\cap T_j\neq\emptyset} 2\theta_{ij}\Big] \\
&\geq \sum_{i=1}^{h}(h_i+1) \;-\; \sum_{j=1}^{t}\beta_j\left[x(\delta(T_j))-2\right]
\end{aligned}
$$

$\diamondsuit$

8 LADDER INEQUALITIES

Ladder inequalities were introduced by Boyd and Cunningham (1991). Boyd et al. (1993), proved that they are facet inducing for the Traveling Salesman Polytope. They can be interpreted in the same way as the inequalities studied in the previous sections by using an extension of the lifting procedure given in Section 3. This extension corresponds to using additional lifting terms, different from slack variables of subtour elimination inequalities.

Let $\mathcal{H} = \{H_1, H_2\}$, $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$, $t \geq 4$ even, $\beta_1, \ldots, \beta_j, \ldots, \beta_t$ be integers associated to the teeth such that (see Figure 10, where the coefficient of the teeth are beside them):

$$
\begin{aligned}
&H_1 \cap H_2 = \emptyset && (8.1)\\
&H_1 \cap T_1 \neq \emptyset, H_2 \cap T_1 = \emptyset && (8.2)\\
&H_2 \cap T_2 \neq \emptyset, H_1 \cap T_2 = \emptyset && (8.3)\\
&T_1 \setminus H_1 \neq \emptyset, T_2 \setminus H_2 \neq \emptyset && (8.4)\\
&T_j \cap H_i \neq \emptyset \qquad \text{for } i = 1,2, \quad \text{for } j \geq 3 && (8.5)\\
&\text{If } T_j \setminus (H_1 \cup H_2) \neq \emptyset,\ \beta_j = 1 \text{ else } \beta_j = 2, \quad \text{for } j \geq 3 && (8.6)\\
&\beta_1 = \beta_2 = 1 && (8.7)
\end{aligned}
$$

The ladder inequality is:

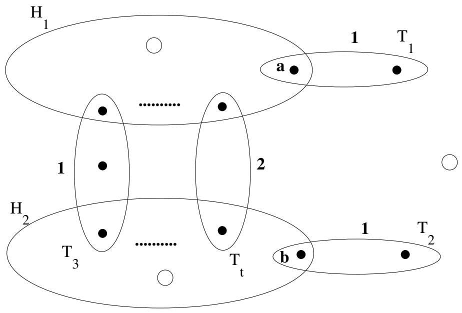

Figure 10: Example of a ladder $(t = 4)$

$$\sum_{i=1}^{2} x(\delta(H_i)) + \sum_{j=1}^{t} \beta_j x(\delta(T_j)) - 2x(H_1 \cap T_1 : H_2 \cap T_2)$$

$$\geq 2t + 2\sum_{j=1}^{t} \beta_j \qquad (8.8)$$

which can be rewritten as:

$$\sum_{i=1}^{2} x(\delta(H_i)) \quad \geq \quad 2t - \sum_{j=1}^{t} \beta_j \big[x(\delta(T_j)) - 2\big] + 2x(H_1 \cap T_1 : H_2 \cap T_2) \quad (8.9)$$

Note the special correcting (or lifting) term $2x(H_1 \cap T_1 : H_2 \cap T_2)$ on the right hand side of inequality 8.9.

For reasons that will become obvious in a moment the authors prefer the following equivalent conditions obtained by replacing H_2 by its complement in V (see Figure 11). Interval of handles relative to a tooth is as defined in Section 6 for Star inequalities, but here teeth can have only 1 or 2 intervals, the first case occurs when a tooth has no node outside any handle.

$$H_1 \subset H_2 \qquad (8.10)$$

$$H_1 \cap T_1 \neq \emptyset, \ H_2 \supset T_1 \qquad (8.11)$$

$$H_2 \cap T_2 \neq \emptyset, \ H_1 \cap T_2 = \emptyset \qquad (8.12)$$

$$T_1 \setminus H_1 \neq \emptyset, \ T_2 \setminus H_2 \neq \emptyset \qquad (8.13)$$

$$T_j \cap H_1 \neq \emptyset, \ T_j \setminus H_2 \neq \emptyset \qquad \text{for all } j \geq 3 \quad (8.14)$$

$$\text{If } T_j \text{ has 2 intervals then } \beta_j = 1 \text{ else } \beta_j = 2 \qquad \text{for all } j \geq 3 \quad (8.15)$$

$$\beta_1 = \beta_2 = 1 \qquad (8.16)$$

Which yields the following form of Inequality 8.9:

$$\sum_{i=1}^{2} x(\delta(H_i)) \quad \geq \quad 2t \quad - \quad \sum_{j=1}^{t} \beta_j \big[x(\delta(T_j)) - 2\big] + 2x(H_1 \cap T_1 : T_2 \setminus H_2) (8.17)$$

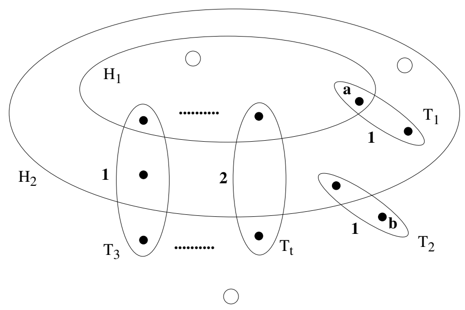

Figure 11: Example of a ladder with nested handles $(t = 4)$

Without the "correcting term", the explanation of the RHS and of the coefficients β_j's goes exactly like for the path inequalities. In fact it is then a special case of Binested inequalities. So the problem is to understand why this correcting term is correct.

Another natural question that arises is why only one tooth in "position" T_1 and only one in "position" T_2? Could we have instead an odd number of them? Without the correcting term the inequality is valid since it is again a special case of binested inequalities. Therefore we are led to the following question:

Question 8.1 *Is the corresponding correcting term: $2\sum_{i,j} x(T_i \cap H_1 : T_j \cap H_2)$, where the summation is on all teeth T_i in the position of T_1 and all teeth T_j in the position of T_2 in the definition of ladder inequalities (see Figure 10), still valid?*

Finally, in view of Figure 11:

Question 8.2 *Can we have more than 2 handles? How general can we make these inequalities?*

Once we understand why this correcting term is correct we will get an easy answer to the other questions.

As already mentioned, the RHS of 8.9 (resp. 8.17) can easily be understood if you consider hamiltonian cycles for which all teeth are tight. Each handle must then have at least t edges of those cycles in its coboundary because $t-1$ is odd. Take an edge $e^\star \in (T_1 \cap H_1 : T_2 \cap H_2)$ (resp. of $(T_1 \cap H_1 : T_2 \setminus H_2)$), that is an edge with one extremity in position "a" and the other in position "b" of Figures 10 and 11. Let Γ be a hamiltonian cycle containing that edge. This cycle cannot have $\mid \Gamma \cap \delta(T_j) \mid = 2$ for all $j = 1, \ldots .t$ together with $\mid \Gamma \cap \delta(H_i) \mid = t$ for all i, else it would not be connected (see Figure 12 where there is no way to connect c to d without crossing the border of H_2 two more times). Therefore for all hamiltonian cycles Γ containing edge $e^\star$, either $\mid \Gamma \cap \delta(H_i) \mid \geq t+2$ for some i =1 or 2, or $\mid \Gamma \cap \delta(T_j) \mid \geq 4$ for some j, without any "gain" on the coboundaries of the handles. Gains on the coboundaries of the handles can only be made at the expense of more teeth having 4 or more edges of their coboundary in Γ. We just proved that no hamiltonian cycle containing $e^\star$ can satisfy Inequalities 8.9 or 8.17 with equality if the correcting term is deleted. Therefore the face it defines is contained in the facet $x_e^\star \geq 0$. With the correcting term the cycles shown in Figure 13 satisfy the inequalities with equality.

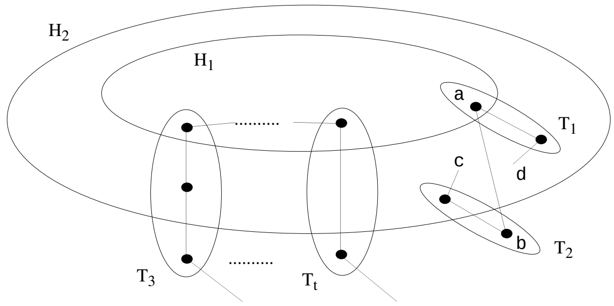

Figure 12: Not every tooth can be tight and every handle have t edges out

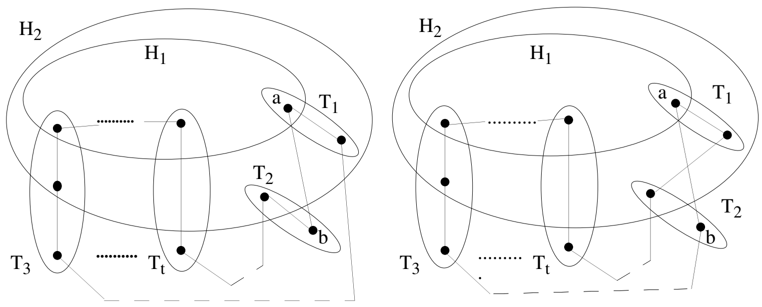

Figure 13: Tight hamiltonian cycles

What happens if we have, say 3 teeth in position T_1? Figure 14 shows a hamiltonian cycle which satisfies to equality the correspondent to Inequalities 8.9 and 8.17 without the correcting term (Note that here the independent term at the RHS has value 10). Therefore the inequality would not be valid with that term since the cycle of Figure 14 contains an edge which would give that correcting term a value of 2.

What about more nested handles as in Figure 15 (with $t-h$ even), but with only one tooth, for each handle, not intersecting properly any other one?

The support $\mathcal{S}$ in Figure 15 would give an inequality of the form

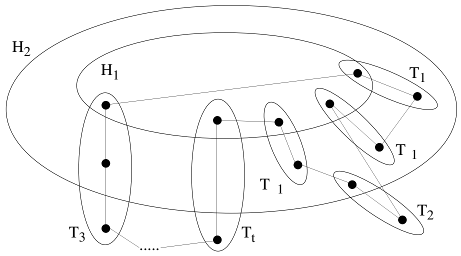

Figure 14: More teeth cancel the correcting term

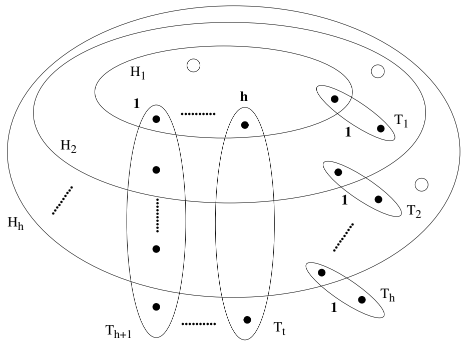

Figure 15: A generalization of ladders

$$\sum_{i=1}^{h} x(\delta(H_i)) \geq h(t-h+2) \quad - \quad \sum_{j=1}^{t} \beta_j \left[x(\delta(T_j)) - 2\right] \quad + \epsilon(\mathcal{S}) \quad (8.18)$$

which is a binested inequality plus a correcting term $\epsilon(\mathcal{S})$.

Repeating the previous argument it is easy to convince oneself that there is no hamiltonian cycle containing an edge from one of the sets $(H_i \cap T_i : T_j \setminus H_j)$ for $j > i$, that can be tight for the corresponding inequality.

Therefore, the most obvious correcting term $\epsilon(\mathcal{S})$ should be: $2\sum_{i,j,i<j} x(H_i \cap T_i : T_j \setminus H_j)$. Unfortunately this does not work as seen in the case of three handles with the cycle shown in Figure 16. The RHS of inequality 8.18 without correcting term is $h(t-h+2) - 2 = 10$. Its LHS for this tour is 12. If the above correcting term was correct, it would have a value of $+4$ since 2 edges are used from the sets that create this term. This yield a value of 14 at the RHS and the inequality is not valid.

So on one hand we know there must be a correcting term, but it is not what is expected.

Question 8.3 *What is the correcting term when we have more than one handle?*

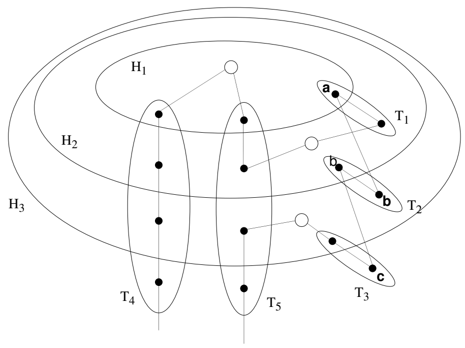

Figure 16: A counterexample with 3 handles

It seems that one can define several facet defining inequalities from this configuration of handles and teeth. Sequential lifting procedures such as the one described in Naddef and Rinaldi (1991), or such as the one obtained by generalizing our procedure to include correcting terms, could be used to find them.

9 SEQUENCE DEPENDENT LIFTING

So far we have been able to define the coefficients of the teeth individually, that is all other teeth except the one considered were forced to stay tight. In other words, we have computed the lifting coefficient associated to each tooth as if the tooth was lifted first. This in particular assumes that the coefficients are independent on the order in which they are computed. As we will see from the following example of Figure 17, which has been provided to us by Maurice Queyranne and which is facet inducing for the TSP polytope, this is not always the case. Here, the lifting coefficients are defined as in Theorem 3.1, but their values are sequence dependent.

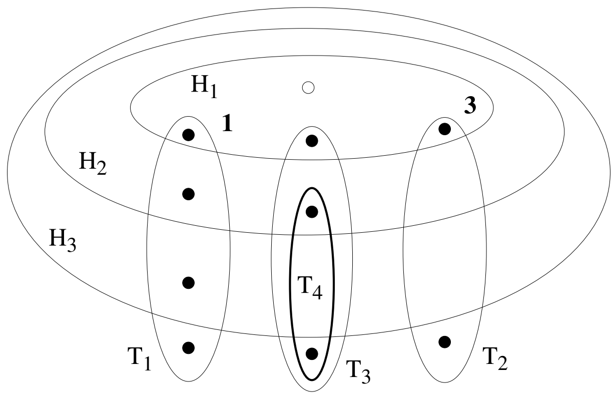

Figure 17: Example of a facet defining inequality

The support in Figure 17 would give an inequality of the form

$$\sum_{i=1}^{3} x(\delta(H_i)) \geq 12 \; - \; \sum_{j=1}^{4} \beta_j \left[x(\delta(T_j)) - 2 \right] \tag{9.1}$$

If you authorize a hamiltonian cycle to intersect the coboundary of T_4 in 4 or more edges restricting all the other teeth to be tight, we cannot compensate by "saving" anything on the coboundaries of the handles. We therefore cannot hope with this procedure to produce anything which would not be just the addition of an inequality obtained without that tooth and the subtour elimination inequality associated to that tooth. In other words, lifting T_4 first gives $\beta_4 = 0$.

The coefficients of T_1 and T_2 can be defined in the manner used so far. This gives $\beta_1 = 1$ and $\beta_2 = 3$. Let us authorize a hamiltonian cycle to intersect the coboundary of T_3 in 4 edges, imposing T_4 to be tight. We can only save 2 edges on the coboundaries of the handles (whether or not we impose T_1 and T_2 to be tight) (see Figure 18a). These 2 edges we save are 2 of the 4 that crossed the border of H_1 when we imposed all teeth to be tight. This yields a coefficient of $\beta_3 = 1$ for T_3, instead of a coefficient of 2 in the Star inequality made up of the same handles and teeth T_1, T_2 and T_3. Now that the coefficients of T_1, T_2 and T_3 are known we can define the coefficient of T_4.

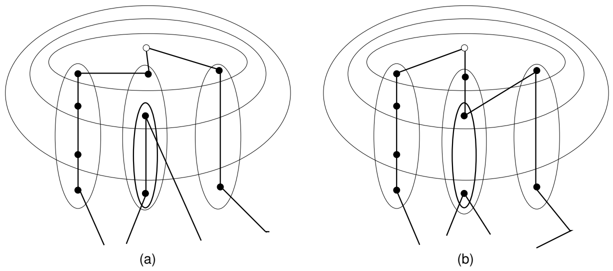

Figure 18: Some traces of hamiltonian cycles

The trace of a tour shown in Figure 18b, shows that we can save 4 edges on the coboundaries of the handles by enabling a cycle to intersect the coboundary of T_4 in 4 edges. Two of these 4 edges are compensated for, by the extra 2 edges in the coboundary of T_3, the 2 others by the extra 2 edges in the coboundary of T_4, which yields a coefficient of $\beta_4 = 1$ for tooth T_4.

10 CONCLUSIONS

In this paper we have explored large families of inequalities which are defined on sets of subsets of nodes which partition into handles and teeth. The rationale behind these inequalities is that either by a parity argument or by the presence of nodes in key positions, when the teeth are tight, the handles are forced to have more edges in their coboundaries than expected. Recently, several authors (Cockburn (2000),Boyd et al. (2000),Naddef (2001)), following Letchford (2000), have a studied inequalities also defined on handles and teeth, but for which parity and position of nodes are not the only arguments. For some teeth containing other teeth, there is a penalty on some edges not crossing the boundaries of handles such that if one does not want to use any of these, one must use an edge of the coboundary of a handle. For more details see for example Naddef (2001).

We hope to have made the Traveling Salesman Polytope more understandable by explaining the rationale behind some of the known inequalities. Some other valid inequalities are easily understandable, such as the hypomatchable inequalities. There is one inequality which has resisted to the search of any explanation: the *Chain Inequalities* defined by Padberg and Hong (1980). We prove here that a generalized version is valid, but no intuitive reason of validity is known so far.

The *generalized chain* inequalities are defined by $\mathcal{H} = \{H\}$, $\mathcal{T} = \{T_1, \ldots, T_j, \ldots, T_t\}$, $\mathcal{S} = \{S_1, \ldots, S_j, \ldots, S_s\}$ and $\mathcal{R} = \{R_1, \ldots, R_j, \ldots, R_r\}$, $s \geq 2, r \geq 2, t \geq 2$ such that:

$$\{H, \{T_i : i = 1, \cdots, t\}\} \text{ satisfy Conditions} \tag{10.1}$$
$$\text{4.1, 4.2 and 4.3 of the combs} \tag{10.2}$$
$$S_j \subset H \qquad \text{for all } j = 1, \cdots, s \tag{10.3}$$
$$R_j \subset V \setminus H \qquad \text{for all } j = 1, \cdots, r \tag{10.4}$$
$$S_j \cap T_i = \emptyset \qquad \text{for all } i \text{ and } j \tag{10.5}$$
$$R_j \cap T_i = \emptyset \qquad \text{for all } i \text{ and } j \tag{10.6}$$
$$t + s + r \text{ is even} \tag{10.7}$$

That is $\mathcal{H}$ and $\mathcal{T}$ satisfy the same conditions as for Comb inequalities except for the parity of t which is replaced by a parity condition on t+s+r. The subsets S_j are contained in the handle and the sets R_j do not intersect it. None of these sets intersect the teeth.

The generalized chain inequality is:

$$x(\delta(H)) + \sum_{j=1}^{t} x(\delta(T_j)) + \sum_{j=1}^{s} x(\delta(S_j)) + \sum_{j=1}^{r} x(\delta(R_j))$$
$$-2 \sum_{i=1}^{s} \sum_{j=1}^{r} x(S_i : R_j) \geq 3t + s + r + 2 \qquad (10.8)$$

Theorem 10.1 *The generalized chain Inequality 10.8 is valid.*

Proof: Let LHS be the left hand side of the Inequality 10.8. We have:

$$LHS \geq 1/2 \Bigg[\sum_{j=1}^{t} x(\delta(T_j \cap H)) + \sum_{j=1}^{t} x(\delta(T_j \setminus H)) + \sum_{j=1}^{t} x(\delta(T_j))$$
$$+ \sum_{j=1}^{s} x(\delta(S_j)) + \sum_{j=1}^{r} x(\delta(R_j)) + x(\delta((\bigcup_{j=1}^{s} S_j) \cup (\bigcup_{j=1}^{r} R_j))) \Bigg]$$
$$\geq 3t + s + r + 1 \qquad (10.9)$$

If t+s+r is even, then 3t + s + r + 1 is odd, and therefore we can add 1 to obtain the RHS of our inequality since the LHS is even. ♢

The Chain inequalities defined in Padberg and Hong (1980) are the special case where $r = s$ and all sets S_j and R_j consist of a single node. In this case t must be even (condition not stated in Padberg and Hong (1980)). What do they say in term of hamiltonian cycles?

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