Matrices st-monotones et polynomes

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Goal

Compute a good st-bound of the steady state probability distribution of a discrete-time Markov chain P

Transform the matrix P to keep the same steady-state distribution and then compute de bounds by Vincent's algorithm

1) $t(P) = (1 - \delta)I + \delta P$

2) $t(P) =$ a polynomial with positive coefficients which adds up to 1

Definition 1 *Let X and Y be random variables taking values on the finite state space $\{1, 2, \ldots, n\}$. Let p and q be probability distribution vectors such that*

$$p_j = Pr(X = j) \quad and \quad q_j = Pr(Y = j) \quad for \;\; j = 1, 2, \ldots, n.$$

Then X is said to be less than Y in the strong stochastic sense, that is, $X \leq_{st} Y$ iff

$$\sum_{j=k}^{n} p_j \leq \sum_{j=k}^{n} q_j \quad for \;\; k = 1, 2, \ldots, n.$$

Theorem 1 *Let P and Q be stochastic matrices respectively characterizing time-homoge- neous MCs $X(t)$ and $Y(t)$. Then* $\{X(t),\ t \in \mathcal{T}\} \leq_{st} \{Y(t),\ t \in \mathcal{T}\}$ *if*

- $X(0) \leq_{st} Y(0)$,
- *st-monotonicity of at least one of the matrices holds, that is,*

  $$\textit{either} \quad P_{i,*} \leq_{st} P_{j,*} \quad \textit{or} \quad Q_{i,*} \leq_{st} Q_{j,*} \quad \forall i, j \quad \textit{such that} \;\; i \leq j,$$

- *st-comparability of the matrices holds, that is,* $P_{i,*} \leq_{st} Q_{i,*}\ \ \forall i$.

Here $P_{i,*}$ refers to row i of P.

Vincent's Algorithm

Algorithm 1 Construction of optimal st-monotone upper bounding DTMC Q corresponding to DTMC P of order n:

$$q_{1,n} = p_{1,n};$$
for $i = 2, 3, \ldots, n$,
$$q_{i,n} = \max(q_{i-1,n}, p_{i,n});$$
for $l = n-1, n-2, \ldots, 1$,
$$q_{1,l} = p_{1,l};$$
for $i = 2, 3, \ldots, n$,
$$q_{i,l} = \max(\textstyle\sum_{j=l}^{n} q_{i-1,j}, \sum_{j=l}^{n} p_{i,l}) - \sum_{j=l+1}^{n} q_{i,j};$$

Property 1 *Let U be another st-monotone upper bounding DTMC for P. Then Q is optimal in the sense that $Q \leq_{st} U$.*

Example 1

$$P = \begin{pmatrix} 0.2 & 0 & 0.3 & 0.5 \\ 0.1 & 0 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.1 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.1 \end{pmatrix}$$

Steady state distribution : $\pi_P = [0.2686, 0.1688, 0.2922, 0.2704]$

Algorithm 1 to P yields the st-monotone upper bounding DTMC

$$Q = \begin{pmatrix} 0.2 & 0 & 0.3 & 0.5 \\ 0.1 & 0 & 0.4 & 0.5 \\ 0.1 & 0 & 0.4 & 0.5 \\ 0.1 & 0 & 0.4 & 0.5 \end{pmatrix}.$$

Steady state distribution : $\pi_Q = [0.1111, 0.0000, 0.3889, 0.5000]$

$$R = \begin{pmatrix} 0.6 & 0 & 0.15 & 0.25 \\ 0.05 & 0.5 & 0.3 & 0.15 \\ 0.2 & 0.15 & 0.55 & 0.1 \\ 0.15 & 0.15 & 0.15 & 0.55 \end{pmatrix},$$

which has the same steady state probability distribution as P, yields the st-monotone upper bounding DTMC

$$S = \begin{pmatrix} 0.6 & 0 & 0.15 & 0.25 \\ 0.05 & 0.5 & 0.2 & 0.25 \\ 0.05 & 0.3 & 0.4 & 0.25 \\ 0.05 & 0.25 & 0.15 & 0.55 \end{pmatrix}.$$

Steady state distribution of $S : \pi_S = [0.1111, 0.3110, 0.2207, 0.3571]$, and it is clearly a better st upper bound on π_P than π_Q.

$$t(P) = (1-\delta)I + \delta P$$

Property 2

$$\pi_{t(P)} = \pi_P$$

Definition 2 *A stochastic matrix is said to be row diagonally dominant (RDD) if all of its diagonal elements are greater than or equal to 0.5.*

Property 3 *Let P be a RDD DTMC, let $R = t(P)$ and let Q ans S be the st-monotone upper bounding DTMC computed by Algorithm 1 for P and R. Then:*

$$\pi_Q = \pi_S$$

Result

Theorem 2 *Let P be a DTMC of order n and Q be the corresponding st-monotone upper bounding DTMC computed by Algorithm 1. Consider the transformation in equation for $\delta \in (0, 1)$, and let S be the st-monotone upper bounding DTMC for R computed by Algorithm 1. Then:*

$$\pi_S \leq_{st} \pi_Q$$

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Let $\mathcal{B}$ be the set of DTMCs of order n, and let $P \in \mathcal{B}$.

(i) t is the operator corresponding to the transformation :

(ii) r is the summation operator used in st comparison:

$$r(P)[i,j] = \sum_{k=j}^{n} P[i,k].$$

Let $\mathcal{A}$ be the set of matrices defined by $r(P)$, where $P \in \mathcal{B}$.

(iii) v is the following operator which transforms $P \in \mathcal{B}$ to a matrix in $\mathcal{A}$:

$$v(P)[i,j] = \begin{cases} \sum_{k=j}^{n} P[1,k], & i = 1 \\ \max(v(P)[i-1,j], \sum_{k=j}^{n} P[i,k]), & i > 1 \end{cases}.$$

Property 4 *The operator corresponding to Vincent's Algorithm is* $r^{-1}v$.

Le résultat de RAIRO 2003 peut s'écrire:

$$r^{-1}vt(P) \leq_{st} tr^{-1}v(P)$$

Property 5 $r^{-1}vt(P) \leq_{st} tr^{-1}v(P)$ *is equivalent to* $vt(P) \leq rtr^{-1}v(P)$, *where the latter comparison is element-wise.*

Property 6 *Unrolling* v *yields the simpler representation*

$$v(P)[i,j] = \max_{m \leq i}(\sum_{k \geq j} P[m,k]).$$

(max, +)...

Generalization

Definition 3 *Let $\mathcal{D}$ be the set of polynomials $\Phi()$ such that $\Phi(1) = 1$, Φ different of Identity, and all the coefficients of Φ are non negative.*

Lemma 1 *$\Phi(P)$ is ergodic and has the same steady-state distribution as P*

Fundamental Lemmas

Lemma 2 *Let $M = r^{-1}v(P)$ then M is the smallest st-monotone stochastic matrix larger than P.*

i.e. let Q be an arbitrary stochastic matrix such that

- *Q is st-monotone*
- *$P <_{st} Q$*

thus M is smaller than Q, i.e. $M <_{st} Q$

Lemma 3 *Let Φ be an arbitrary polynomial in $\mathcal{D}$, let P et Q be two stochastic matrices in $\mathcal{A}$, if $P <_{st} Q$ and Q st-monotone then*

1. *$\Phi(Q)$ is st-monotone*
2. *$\Phi(P) <_{st} \Phi(Q)$*

Theorem

Theorem 3 *Let Φ be an arbitrary polynomial in $\mathcal{D}$, we have*

$$r^{-1}v\Phi(P) <_{st} \Phi(r^{-1}v(P))$$

Property 7 *Consider an arbitrary polynomial $\Phi()$ in $\mathcal{D}$ for an arbitrary ergodic Markov chain P, we have :*

$$\pi_P <_{st} \pi_{r^{-1}v\Phi(P)} <_{st} \pi_{r^{-1}v(P)}$$

Proof

applying lemma 3, with $Q = r^{-1}v(P)$, we get
$\Phi(P) <_{st} \Phi(r^{-1}v(P))$. And this last matrix is st-monotone.

Now, I consider the aplication of operator $r^{-1}v$ to matrix $\Phi(P)$. According to Lemma 2 $r^{-1}v(\Phi(P))$ is the best st-monotone matrix greater than $\Phi(P)$. As $\Phi(r^{-1}v(P))$ is another st-monotone upperbound, one must have

$$r^{-1}v\Phi(P) <_{st} \Phi(r^{-1}v(P))$$

And the proof is completed.

Example

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.4 & 0 \\ 0.2 & 0.1 & 0.3 & 0.4 \end{pmatrix}$$

$\phi(X) = X/2 + 1/2 \ \psi(X) = X^2/2 + 1/2.$

$$\phi(P) = \begin{pmatrix} 0.55 & 0.1 & 0.2 & 0.15 \\ 0.1 & 0.65 & 0.1 & 0.15 \\ 0.05 & 0.25 & 0.7 & 0 \\ 0.1 & 0.05 & 0.15 & 0.7 \end{pmatrix} \quad \psi(P) = \begin{pmatrix} 0.575 & 0.155 & 0.165 & 0.105 \\ 0.08 & 0.63 & 0.155 & 0.135 \\ 0.075 & 0.185 & 0.65 & 0.09 \\ 0.075 & 0.13 & 0.17 & 0.625 \end{pmatrix}$$

Then, we apply operators v and r^{-1} to obtain the bounds :

$$r^{-1} v\phi(P) \begin{pmatrix} 0.55 & 0.1 & 0.2 & 0.15 \\ 0.1 & 0.55 & 0.2 & 0.15 \\ 0.05 & 0.25 & 0.55 & 0.15 \\ 0.05 & 0.1 & 0.15 & 0.7 \end{pmatrix} r^{-1} v\psi(P) \begin{pmatrix} 0.575 & 0.155 & 0.165 & 0.105 \\ 0.08 & 0.63 & 0.155 & 0.135 \\ 0.075 & 0.185 & 0.605 & 0.135 \\ 0.075 & 0.13 & 0.17 & 0.625 \end{pmatrix}$$

$$r^{-1}v(P) = \begin{pmatrix} 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$$

Finally, we compute the st-st distribution for the bounds and the initial matrix :

$$\pi_P = (0.1530, 0.3025, 0, 3167, 0.2278)$$

$$\pi_{r^{-1}v(P)} = (0.1, 0.2, 0, 3667, 0.3333)$$

$$\pi_{r^{-1}v\phi(P)} = (0.1259, 0.2587, 0, 2821, 0.3333)$$

$$\pi_{r^{-1}v\psi(P)} = (0.1530, 0.2997, 0, 2916, 0.2557)$$

Clearly, bounds obtained by polynomial ψ are better than former bounds (polynomial ϕ and continuous-time version of matrix P).