Institut National de Recherche en Informatique et Automatique

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Robustesse des techniques de Monte Carlo dans l'analyse d'événements rares

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Outline

1 - Generalities : ARMOR's work on Monte Carlo

- focus on the rare event case
- static and dynamic (Markov) models
- in the dynamic case, transient and stationary analysis
- algorithm design and generic analysis of Monte Carlo techniques

More precisely :

(i) (static) reliability analysis (network reliability area),

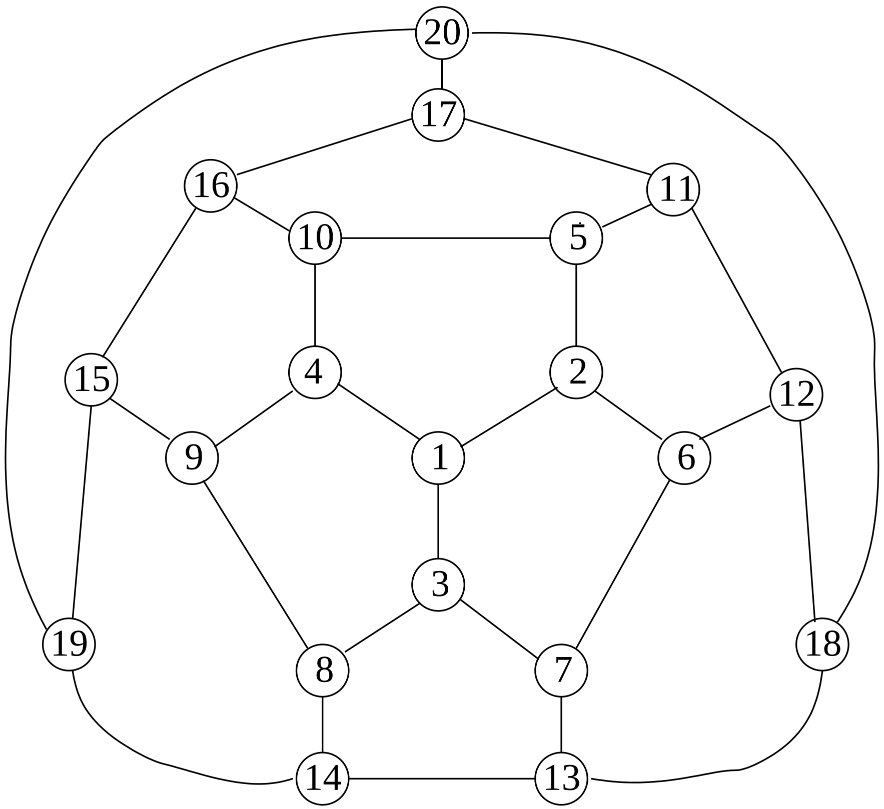
(ii) transient analysis of Markov models : estimation of the reliability at time t,

(iii) transient analysis of Markov models : Importance Sampling applied to estimating the MTTF of a system,

(iv) theoretical analysis of properties of Monte Carlo evaluation methods : illustration in the case of static models.

and variations around (performability, . . .).

Example of (i) :



– a communication network
– nodes are perfect, lines (edges) can fail
– lines are up or down, independently
– nodes 1 and 20 must talk to each other
– network reliability R is the probability that there is at least, a path up between 1 and 20.

Example of (ii), (iii) :

- a multi-component system (say, a database),
- K types of components (disks, power units, servers, . . .),
- type-k components fail with rate λ_k, possible failure propagations
- reparations following an exponential distribution
- system is up iff at least n_k type-k components are,
- we get a Markov model, homogeneous, finite, with states classed up or down.

- An interesting metric is

MTTF = mean time to system down.

- A more detailed metric is the reliability at time t,

$$R(t) = \Pr(\text{system is up from time 0 to time } t).$$

2 - Standard Monte Carlo

- Consider a real random variable X and a real function $\psi()$.
- Let $\gamma = \mathsf{E}(\psi(X))$ and $\sigma^2 = \mathsf{Var}(\psi(X))$.
- The standard estimator of γ :
  - build n independent copies $X_1, \cdots, X_n$ of X
  - return $\gamma_n^{\text{STD}} = \dfrac{1}{n}\sum_{i=1}^{n}\psi(X_i)$.
- $\mathsf{E}(\gamma_n^{\text{STD}}) = \gamma, \quad \mathsf{Var}(\gamma_n^{\text{STD}}) = \sigma^2/n$

- A centered confidence interval for γ with (confidence) level δ is

$$\mathsf{C}_n^{\mathsf{STD}} = \left(\gamma_n^{\mathsf{STD}} \pm z_\delta \sqrt{\mathsf{Var}(\gamma_n^{\mathsf{STD}})}\right)$$

– the meaning is $\Pr(\mathsf{C}_n^{\mathsf{STD}} \ni \gamma) = \delta$

– factor z_δ is

$$z_\delta = \mathcal{N}^{-1}\left(\frac{1+\delta}{2}\right),$$

where

$$\mathcal{N}(x) = \frac{1}{2\pi}\int_0^x \exp(-u^2/2)d\,u.$$

(z_δ is the $1-\delta/2$ quantile of the std. normal distr.).

3 - Rarity

Rarity happens when $\gamma \ll 1$.

Examples :

- In the network reliability problem,

$$X = 1_{\{\text{chosen nodes can not communicate}\}},$$

and

$$\psi(x) = x.$$

(That is, $\gamma = 1 - R$).

When network components are very reliable, $\gamma \ll 1$.

- When estimating MTTF in a Markov model, we first write

$$\text{MTTF} = \frac{\mathsf{E}(\min\{U,V\})}{\Pr(U<V)},$$

where U is the r.v. “time to return to 0”, V is the r.v. “time to absorption”, and 0 is the initial state of the chain, assumed to be an “up” state.

- Focus on estimating the denominator, the only difficult component of the formula : $\gamma = \Pr(U < V)$ and we can put this into the general framework as before.
- If the system is highly reliable, again $\gamma \ll 1$.

Rarity parameter

- We assume that X depends on some real parameter ε (and, thus, γ, σ are also functions of ε).
- We call ε a *rarity parameter* because it verifies

$$\lim_{\varepsilon \to 0} \gamma = 0.$$

- Example : in the static model, we can assume that the reliability of line i, r_i, has the form

$$r_i = 1 - a_i \varepsilon^{b_i}.$$

where a_i and b_i are positive constants.

- In this case, we can prove that there exists some $r > 0$ such that

$$\gamma = \Theta(\varepsilon^r).$$

- If we look at the relative error when estimating γ using the standard estimator γ_n^{STD},

$$\lim_{\varepsilon \to 0} z_\delta \frac{\sqrt{\text{Var}(\gamma_n^{\text{STD}})}}{\gamma} = \frac{z_\delta}{\sqrt{n}} \lim_{\varepsilon \to 0} \sqrt{\frac{1-\gamma}{\gamma}} = \infty.$$

This is why, in general, we can not use γ_n^{STD} in case of rare events.

4 - Bounded Relative Error (BRErr)

- Consider now any estimator γ_n of γ (for instance, an estimator built using some Importance Sampling method).
- Assume E$(\gamma_n) = \gamma$ (that is, γ_n is unbiased) and denote $\sigma_n^2 =$ Var(γ_n).
- We can again build a centered confidence interval for γ from γ_n, writing

$$\mathrm{C}_n = (\gamma_n \pm z_\delta \sigma_n)$$

- The **relative error** is

$$\mathrm{RErr} = z_\delta \frac{\sigma_n}{\gamma}.$$

- We say that we have a bounded relative error (BRErr) if RErr remains bounded as $\varepsilon \longrightarrow 0$.
- This desirable property states that the relative size of the confidence interval remains bounded as $\varepsilon \longrightarrow 0$.
- In the case of the standard estimator, $\sigma_n^2 = \gamma(1-\gamma)/n$, and

$$\text{RErr} = z_\delta \frac{\sigma_n}{\gamma} = z_\delta \sqrt{\frac{1-\gamma}{n\gamma}} \longrightarrow \infty \quad \text{when } \varepsilon \longrightarrow 0.$$

4.1 - Asymptotic Optimality

- Often used in queuing theory when γ_n comes from the Importance Sampling method.
- $\gamma = E_f[g(X)] = E_{f^*}[g(X)L(X)]$ where $L(x) = f(x)/f^*(x)$.
- $\hat{\gamma}_n^{IS}$ is called asymptotically optimal, if $\lim_{\varepsilon\to 0} \frac{\ln E_{f^*}[g(X)^2 L(X)^2]}{\ln \gamma} = 2$.
- Bounded Relative Error implies Asymptotic Optimality (recently proved by Sandmann), the reciprocal being false.
- Bounded Relative Error appears to be the right property.

5 - Bounded Normal Approximation (BNA)

- **Berry-Esseen** : if $F_n()$ is the cdf of $(\gamma_n - \gamma)/\hat{\sigma}_n$, where $\hat{\sigma}_n^2$ is the standard estimator of σ^2, we have that for all real x,

$$|F_n(x) - \mathcal{N}(x)| \leq \frac{a\varrho}{\sigma_n^3\sqrt{n}},$$

  with $\varrho = \mathsf{E}(|\psi(X)|^3)$.
- The estimator γ_n has Bounded Normal Approximation iff the ratio ϱ/σ_n^3 remains bounded when $\varepsilon \to 0$
- Proved that BNA implies BRErr, the reciprocal being false. This means again that BNA appears as the right property to look for.

6 - Need for an extension of BRErr : BREff

- What is important in simulation ? the RErr *for a given simulation time*, not for a number n of replications.
- BRErr does not incorporate the second important characteristic of an estimator : *simulation time* (the computational cost).
- The average simulation time to get *one* replication can
  - *increase* with ε, or
  - *decrease* with ε.
- We will illustrate next this last case for a simulation method estimating the reliability of a static stochastic network.

6.1 - Illustration : simulation for static reliability analysis

- Consider the network reliability problem. Denote by G the undirected graph modelling the network, made of M links.
- Computing R, the probability that two fixed nodes s and t are connected, is an NP-hard problem.
- The method :
  - Let $\mathcal{P} = \{P_1, P_2, \cdots, P_H\}$ be a set of elementary (disjoint) paths connecting nodes s and t,
  - Let π_h be the event “all links of path P_h work” and $p_h = \Pr(\pi_h)$.
  - Assume an infinite sequence of independent copies of G is built.
  - Let F be the random variable “first element in the sequence where every path in $\mathcal{P}$ has at least one link that does not work”.

– Variable F is geometrically distributed with parameter $q = \prod_{h=1}^{H}(1 - p_h) = \Pr($no path in $\mathcal{P}$ “works”$)$:

$$\Pr(F = f) = (1 - q)^{f-1} q,$$

and, in particular,

$$\mathsf{E}(F) = \frac{1}{q}.$$

– Idea : sample first from the geometric distribution of F. The estimator is then built assuming that in the first $F - 1$ copies, nodes s and t are connected.

– Assume $F = f$. To know if they are connected in the fth copy, we must sample the network conditioning on the fact that at least one edge in each path is down.

– Consider a path P_h. Call W_h the r.v. giving the first failed edge of P_h.

– Write $P_h = (i_{h,1}, i_{h,2}, \cdots, i_{h,K_h})$.

We have

$$\Pr(W_h = w) = \frac{r_{i_{h,1}} r_{i_{h,2}} \cdots r_{i_{h,w-1}} (1 - r_{i_{h,w}})}{1 - r_{i_{h,1}} r_{i_{h,2}} \cdots r_{i_{h,K_h}}}.$$

– Assume $W_h = w$. Edges $i_{h,1}, i_{h,2}, \cdots, i_{h,w-1}$ are set to “up”, edge $i_{h,w}$ to “down”. Remaining edges in P_h are sampled from their *a priori* Bernoulli distributions.

- We have built (virtually) f copies of G by sampling F once and the network configuration (the states of the M links) once.

- Observe that
  - We have unbounded RErr (same variance than for crude Monte Carlo),
  - We sample F and the network, on the average, nq (= $n/(1/q)$) times, and we test the connectivity between s and t also nq times on the average.
  - The simulation time (or computing cost) is, on the average, $O(nq(M + K_1 + \cdots + K_H))$ (= $O(nqM)$ if we wish). It decreases with ε if $r_i = 1 - a_i\varepsilon^{b_i}$, since q decreases with ε.

6.2 - Bounded relative Efficiency

- It basically addresses the (relative) variance of an estimator obtained during a given simulation time.
- Consider an estimator γ_n of γ, with variance σ_n^2, built from n replications (possibly dependent), and denote by t_n the average simulation time needed to get these n replications.

  The relative efficiency of γ_n is

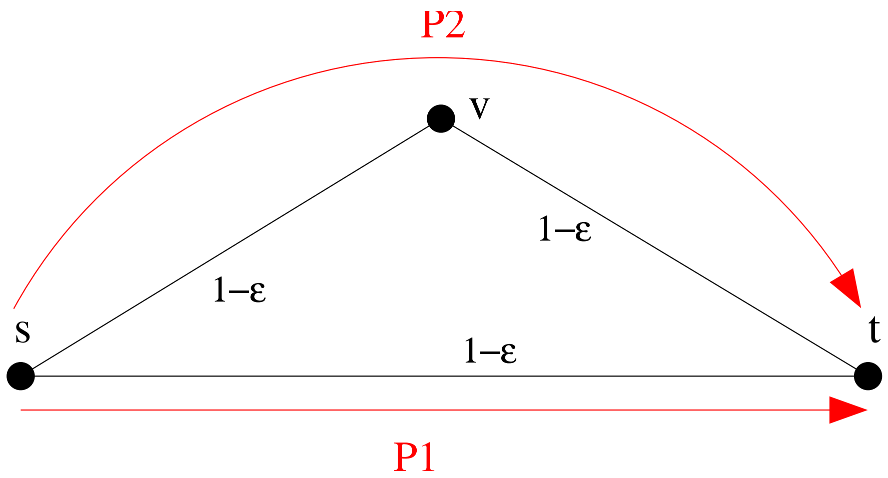
$$\text{REff} = \frac{\gamma^2}{\sigma_n^2 t_n}.$$

  The estimator γ_n has bounded relative efficiency (BREff) if there exists $d > 0$ such that REff is minored by d for all ε.

6.3 - Sufficient condition on the static reliability example

- Let $r > 0$ be the real such that $\gamma = \Theta(\varepsilon^r)$.
- $\forall P_h \in \mathcal{P}$, let $P_h = (i_{h,1}, \cdots, i_{h,K_h})$ and $b_h = \min_{1 \le k \le K_h} b_{i_{h,k}}$ (b_h is the exponent of ε in the most reliable edge of P_h (as $\varepsilon \to 0$)).
- Theorem : the estimator of the static unreliability described in previous section verifies Bounded Relative Efficiency if $\sum_{h=1}^{H} b_h \ge r$.

7 - Illustrations 7.1 - Illustration on a (very) small problem



- $\gamma = 1 - R = \varepsilon^3 + 2\varepsilon^2(1-\varepsilon) \approx 2\varepsilon^2$
- $\sigma^2 = \gamma(1-\gamma) \approx 2\varepsilon^2$
- $\sigma/\gamma \approx 1/(\sqrt{2}\varepsilon)$: no BRErr
- $p_1 = 1 - \varepsilon, p_2 = (1-\varepsilon)^2$
- $q = (1-p_1)(1-p_2) \approx 2\varepsilon^2$;
- t_n proportional to q
- REff $\approx \dfrac{\gamma^2}{\sigma^2 q}$ bounded : BREff.

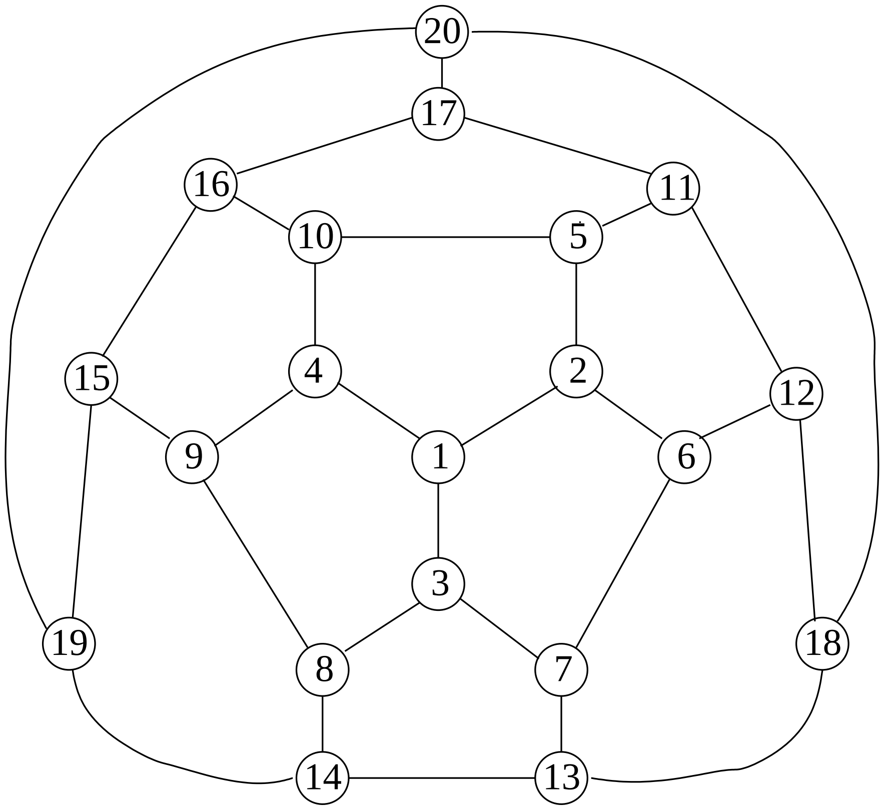
Results on the simple topology, with a number of replications fixed to $n = 10^4$:

$r_i \; \forall i$	Est.	Conf. interval	RErr	KS stat.
0.5	3.7790000e-01	(3.6839622e-01,3.8740378e-01)	2.51489e-02	5.6705e-02
0.9	1.9012678e-02	(1.8992726e-02,1.9032629e-02)	1.04937e-03	4.8610e-02
0.99	2.0000000e-04	(-7.7171997e-05,4.7717200e-04)	1.38586e+00	2.5436e-01
0.999	0	(0, 0)	—	1
0.9999	0	(0, 0)	—	1

Results for a simulation time $T = 10$ seconds :

$r_i \; \forall i$	Est.	Conf. interval	RErr
0.5	3.7498089e-01	(3.7466460e-01,3.7529718e-01)	8.43484e-04
0.9	1.9012678e-02	(1.8992726e-02,1.9032629e-02)	1.04937e-03
0.99	1.9905429e-04	1.9884527e-04,1.9926330e-04)	1.05005e-03
0.999	2.0003614e-06	(1.9982415e-06,2.0024813e-06)	1.05975e-03
0.9999	2.0014432e-08	1.9993102e-08,2.0035762e-08)	1.06572e-03

7.2 - Illustration : efficiency on the dodecahedron topology



with $s = 1$ and $t = 20$

$r_i \ \forall i$	Relative eff. w.r.t. crude MC
0.9	18.9
0.95	188.3
0.98	3800.2

$r_i \ \forall i$	Est.	Conf. interval	RErr	KS stat.
0.5	7.0820e-01	(6.99290e-01,7.17110e-01)	1.25818e-02	3.3896e-02
0.9	3.2000e-03	(2.09298e-03,4.30702e-03)	3.45945e-01	9.6531e-02
0.99	0	(0, 0)	—	8.2840e-01
0.999	0	(0, 0)	—	1

Results for a simulation time $T = 5$ seconds :

$r_i \ \forall i$	Est.	Conf. interval	RErr	KS stat.
0.5	7.1201e-01	(7.0918e-01 , 7.1485e-01)	3.98e-03	4.3126e-02
0.9	2.8894e-03	(2.7776e-03 , 3.0012e-03)	3.87e-02	7.0679e-02
0.99	2.0527e-06	(1.9211e-06 , 2.1844e-06)	6.41e-02	3.3475e-02
0.999	2.0047e-09	(1.8705e-09 , 2.1388e-09)	6.69e-02	6.5625e-02
0.9999	1.9812e-12	(1.8474e-12 , 2.1150e-12)	6.75e-02	5.4364e-02

8 - Generalized Bounded Normal Approximation

- Again, in BNA, time (cost) not taken into account.
- Fix time (computational budget) T and let $n = n(T)$ be the corresponding average number of of iterations.
- The estimator $\gamma_{n(T)}$ verifies GBNA if the ratio $\varrho_{n(T)}/(\sigma^3_{n(T)}\ n(T))$ remains bounded when $\varepsilon \longrightarrow 0$.
- It can be proved now that GNBA implies BREff and that in the case of the static reliability problem, both properties are equivalent.

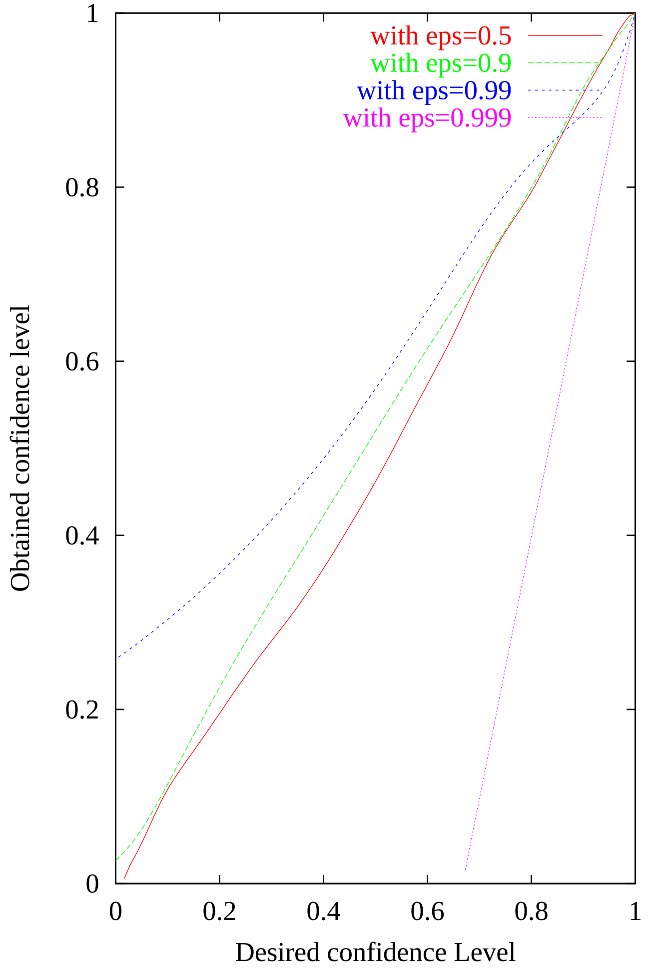
Coverage function

- Confidence interval $R(\eta, \mathbb{X})$ for γ, at confidence level η with (random) data $\mathbb{X}$.
- We should have $\Pr[\gamma \in R(\eta, \mathbb{X})] = \eta$.
- If $\eta^* = \inf\{\eta \in [0,1] : \gamma \in R(\eta, \mathbb{X})\}$, then, η^* should be uniformly distributed :

$$F_{\eta^*}(\eta) = \Pr[\eta^* \leq \eta] = \eta.$$

- $F_{\eta^*}(\eta)$ is the *actual* coverage level : empirical estimation.

Illustration on the simple topology



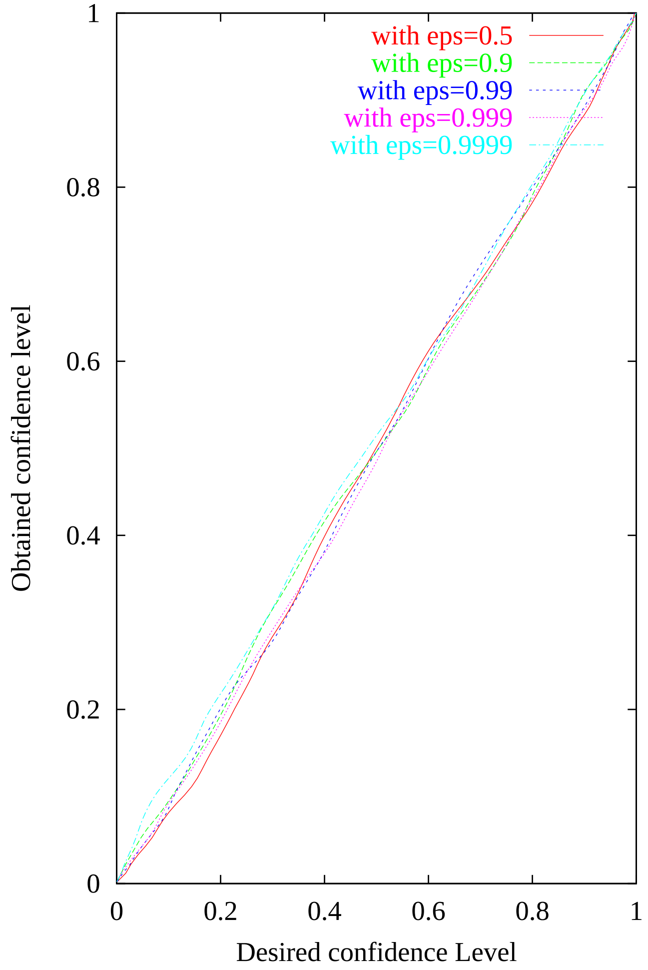
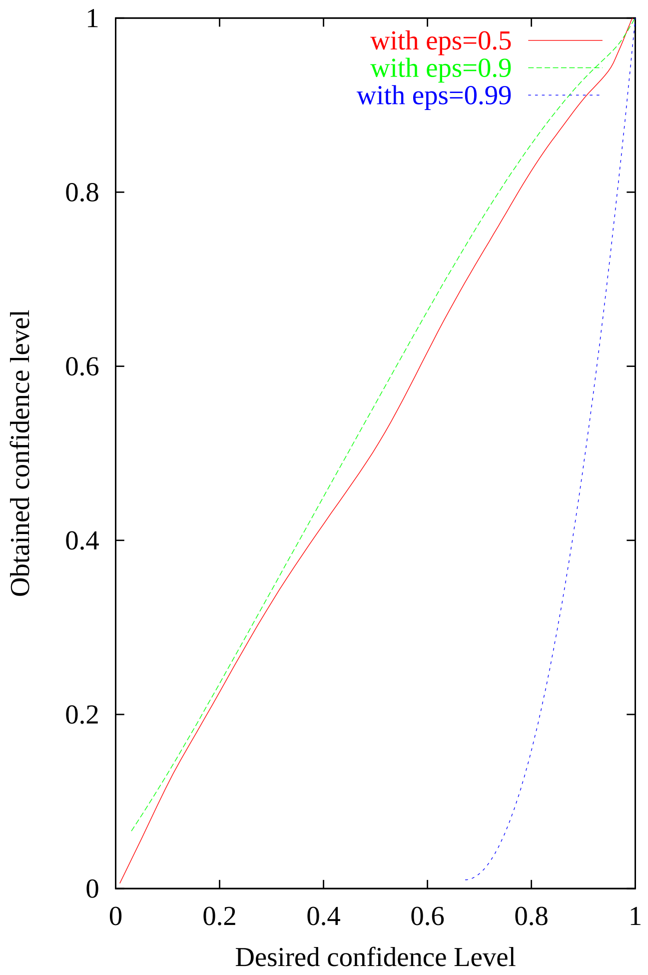
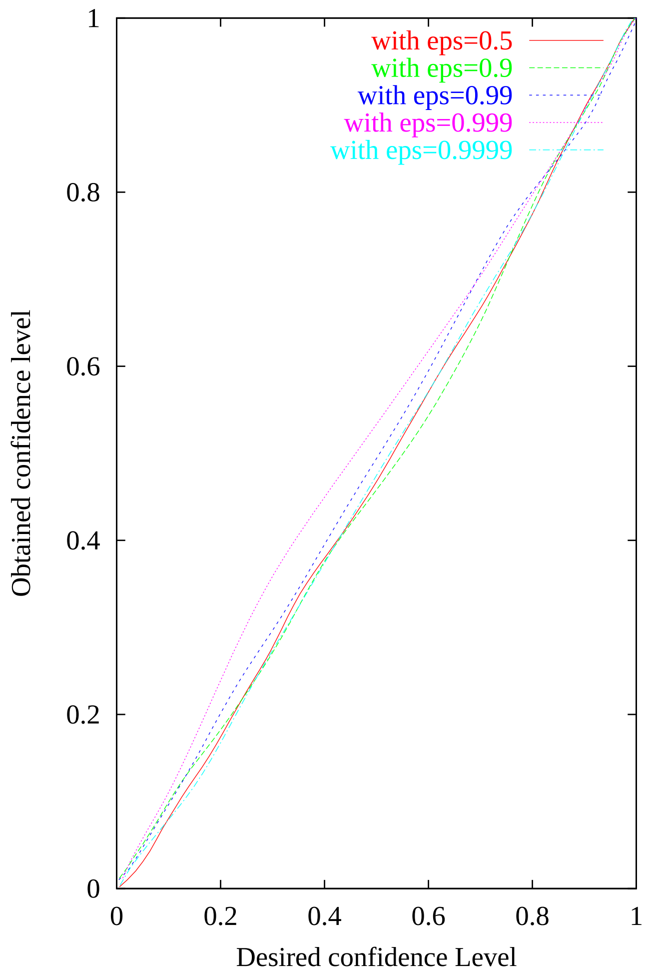


Illustration on the Dodecahedron topology





9 - Conclusions

- Rare event simulation requires sophisticated techniques with robustness properties.
- Bounded Relative Error does not take into account the estimator's full information.
- We have
  - defined Bounded Relative Efficiency to cope with this problem
  - defined Generalized Bounded Normal Approximation (and previously BNA)
  - Illustrated the validity of the approach on a reliability analysis problem.
  - deep investigation of the coverage of the confidence interval