Lecture 3 – Maths for Computer Science
More on Fibonacci numbers and Stern sequence

Denis TRYSTRAM
Lecture notes MoSIG1

Oct. 2023

Recall: various bilinear progressions

Fibonacci sequence

$F(n+1) = F(n) + F(n-1)$ with $F(0) = 1$ and $F(1) = 1$

Lucas' numbers

Same as Fibonacci with a different seed.
$L(n+1) = L(n) + L(n-1)$ with $L(0) = 1$ and $L(1) = 3$

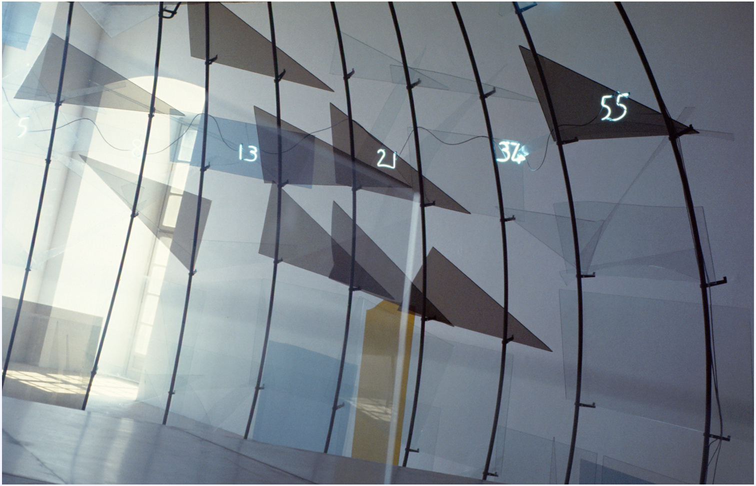
Stern sequence

$s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$ with $d(0) = 0$ and $d(1) = 1$

Derangements

$d(n+1) = n\,(d(n-1) + d(n-2))$ with $d(0) = 1$ and $d(1) = 2$

Fibonacci numbers are everywhere



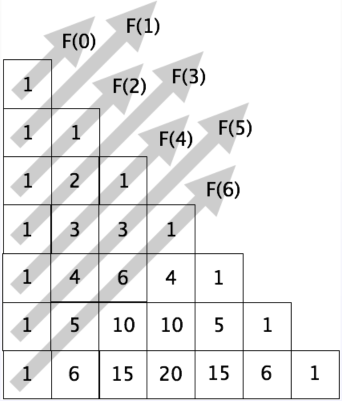
Recall

- Fibonacci numbers are the number of pairs of rabbits that can be produced at the successive generations.
- Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition:
Given the two numbers $F(0) = 1$ and $F(1) = 1$
the Fibonacci numbers are obtained by the following expression:
$F(n+1) = F(n) + F(n-1)$

Link with the Pascal's triangle

Could you prove the following property?



Cassini identity

Proposition:
$F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1}$ for $n \geq 1$

Can we get some intuition on the first ranks?

$n = 1,\ F(0).F(2) = F(1)^2 + 1 = 2$

$n = 2,\ F(1).F(3) = F(2)^2 - 1 = 4 - 1 = 3$

$n = 3,\ F(2).F(4) = F(3)^2 + 1 = 9 + 1 = 10$

$n = 4,\ F(3).F(5) = F(4)^2 - 1 = 25 - 1 = 24$

...

Proof (by induction)

- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n{+}2) = F(n)(F(n{+}1){+}F(n)) = F(n).F(n{+}1){+}F(n)^2$

Proof (by induction)

- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n).F(n+1)+F(n)^2$
  Replace the last term using the recurrence hypothesis:
  $F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1}$
  $= F(n-1).F(n+1) + (-1)^{n+2}$

Proof (by induction)

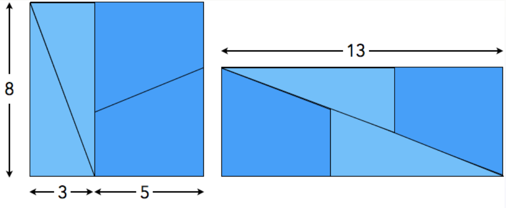
- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n).F(n+1)+F(n)^2$
  Replace the last term using the recurrence hypothesis:
  $F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1}$
  $= F(n-1).F(n+1) + (-1)^{n+2}$
  Thus,
  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$
  $= F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$

Proof (by induction)

- The **basis case** $n = 1$ holds since $F(0).F(2) = F(1)^2 + 1 = 2$.
- The **induction step** is proved assuming the Cassini identity holds at rank n.
  Apply the definition of $F(n+2)$:
  $F(n).F(n+2) = F(n)(F(n+1)+F(n)) = F(n).F(n+1)+F(n)^2$
  Replace the last term using the recurrence hypothesis:
  $F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1}$
  $= F(n-1).F(n+1) + (-1)^{n+2}$
  Thus,
  $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2}$
  $= F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$
  Apply again the definition of Fibonacci sequence
  $F(n) + F(n-1) = F(n+1)$, we obtain:
  $F(n).F(n+2) = F(n+1)^2 + (-1)^{n+2}$

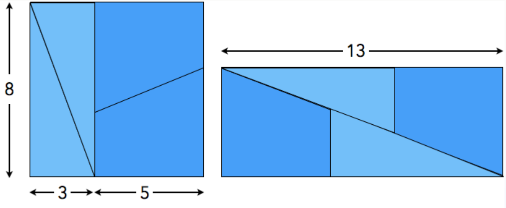
A Paradox (favorite puzzle of Lewis Carroll)

Consider a chess board (8 by 8 square) and cut it into 4 pieces, then reassemble them into a rectangle.



A Paradox (favorite puzzle of Lewis Carroll)

Consider a chess board (8 by 8 square) and cut it into 4 pieces, then reassemble them into a rectangle.



The surface of the square is $F(n)^2$ while the rectangle is $F(n+1).F(n-1)$.

The Cassini identity is applied for $n = 5$, $F(5) = 8$.

- On one side, the surface is $8 \times 8 = 64$
- On the other side $13 \times 5 = 65$

What's wrong?

Explanation

The paradox comes from the representation of the "diagonal" of the rectangle which does not coincide with the hypothenuse of the right triangles of sides $F(n+1)$ and $F(n-1)$.
In other words, it always remains (for any n) an empty space (corresponding to the unit size of the basic square of the chess board).

The greater n, the better the paradox because the *deformation* of the surface of this basic square becomes more tiny.

Computing $F(n)$ fast

$F(n)$ can be computed in $log_2(n)$ steps.

Proposition.

For all integers n:
(a) $F(2n) \ = \ (F(n))^2 \ + \ (F(n-1))^2$
(b) $F(2n+1) \ = \ F(n) \times (2F(n-1) + F(n))$

Details (a) – Proof by induction

The base case $n = 1$ is true because

$$\begin{aligned} F(2) &= (F(1))^2 + (F(0))^2 = 2 \\ F(3) &= F(1) \times (2F(0) + F(1)) = 3 \end{aligned}$$

Assume that the property holds for n, for both $F(2n)$ and $F(2n+1)$.

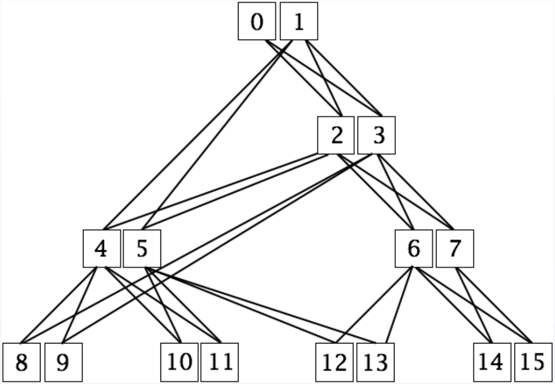
$$F(2(n+1)) = F(2n+1) + F(2n)$$
$$= (F(n))^2 + (F(n-1))^2 + F(n) \times (2F(n-1) + F(n))$$
$$= (F(n))^2 + (F(n-1))^2 + 2(F(n) \times F(n-1)) + (F(n))^2$$
$$= (F(n) + F(n-1))^2 + (F(n))^2$$
$$= (F(n+1))^2 + (F(n))^2$$

Details (b)

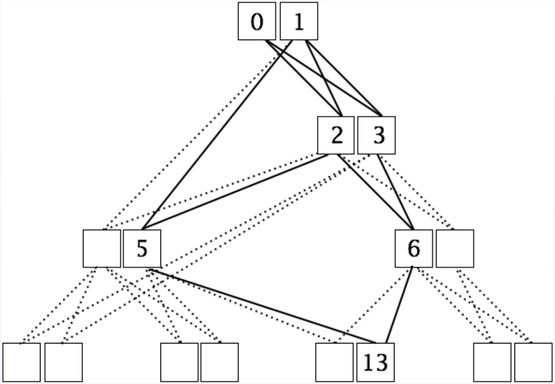
We again start by applying the defining recurrence of the Fibonacci numbers on $F(2(n+1)+1)$

$$= F(2(n+1)) \; + \; F(2n+1)$$
$$= (F(n+1))^2 \; + \; F(n)^2 \; + \; F(n) \times (2F(n-1) + F(n))$$
$$= (F(n+1))^2 \; + \; 2(F(n-1) + F(n)) \times F(n)$$
$$= (F(n+1))^2 \; + \; 2F(n+1) \times F(n)$$

Pictorially



Pictorially (from one node)



Definition of Lucas' numbers

A natural question is:

what happens if we change the first ranks of the sequence keeping the same recurrence pattern?

Definition of Lucas' numbers

A natural question is:

what happens if we change the first ranks of the sequence keeping the same recurrence pattern?

It has been studied by the french mathematician Edouard Lucas, starting at 2 and 1 .

For some reasons that will be clarified later, the sequence is shifted backwards (we take the convention $L(-1) = 2$).

Definition of Lucas' numbers

Definition:
Given the two numbers $L(0) = 1$ and $L(1) = 3$
all the other Lucas' numbers are obtained by the same progression as Fibonacci:

- $L(n+1) = L(n) + L(n-1)$

Definition of Lucas' numbers

Definition:
Given the two numbers $L(0) = 1$ and $L(1) = 3$
all the other Lucas' numbers are obtained by the same progression as Fibonacci:

- $L(n+1) = L(n) + L(n-1)$

n	-1	0	1	2	3	4	5	6	7	8	9	10
F(n)		1	1	2	3	5	8	13	21	34	55	...
L(n)	2	1	3	4	7	11	18	29	47	76	123	...

- There are strong links[1] with Fibonacci numbers.

In particular, we established before that:

$F(n+2) = 1 + \sum_{k=0}^{n} F(k)$.

We have similarly:

$L(n+2) = 1 + \sum_{k=-1}^{n} L(k)$
since the basic step of the induction is still valid[2].
$L(2) = L(-1) + L(0) + 1 = 2 + 1 + 1 = 4$.

[1] of course
[2] It will be true for all the progressions where $u_1 = 1$

A first Property

We can also easily show that the Lucas number of order n is the symmetric sum of two Fibonacci numbers:

Proposition.
$L(n) = F(n-1) + F(n+1)$ for $n \geq 1$

Let *check* this property on the first ranks:

$n = 2,\ L(2) = F(1) + F(3) = 1 + 3 = 4$

$n = 3,\ L(3) = F(2) + F(4) = 2 + 5 = 7$

$n = 4,\ L(4) = F(3) + F(5) = 3 + 8 = 11$

$n = 5,\ L(5) = F(4) + F(6) = 5 + 13 = 18$

...

Proof by induction

- The **basis case** (for $n = 1$) is true since $L(1) = 3 = F(2) + F(0) = 2 + 1$.
- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$

Proof by induction

- The **basis case** (for $n = 1$) is true since
  $L(1) = 3 = F(2) + F(0) = 2 + 1$.
- **Induction step:** Let assume the property holds at all ranks
  $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$

Proof by induction

- The **basis case** (for $n = 1$) is true since $L(1) = 3 = F(2) + F(0) = 2 + 1$.
- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for
  $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$

Proof by induction

- The **basis case** (for $n = 1$) is true since $L(1) = 3 = F(2) + F(0) = 2 + 1$.
- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:
  Apply the definition of Lucas' numbers:
  $L(n+1) = L(n) + L(n-1)$
  Apply the induction hypothesis on both terms:
  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$
  Apply now the definition of Fibonacci numbers for $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$ replace them in the previous expression:
  $L(n+1) = F(n+2) + F(n)$

which concludes the proof.

Extension 1

Notice that using a similar approach, we obtain
$L(n) = F(n+2) - F(n-2)$

What happens if we generalize?

Extension 1

Notice that using a similar approach, we obtain
$L(n) = F(n+2) - F(n-2)$

What happens if we generalize?

Proposition.
$2.L(n) = F(n+3) + F(n-3)$

Extension 1

Notice that using a similar approach, we obtain
$L(n) = F(n+2) - F(n-2)$

What happens if we generalize?

Proposition.

$2.L(n) = F(n+3) + F(n-3)$

Proof.

We start from $L(n) = F(n+2) - F(n-2)$
$F(n+2) = F(n+3) - F(n+1)$ and
$F(n-2) = F(n-1) - F(n-3)$
$L(n) = F(n+3) - (F(n+1) + F(n-1)) + F(n-3)$
$2.L(n) = F(n+3) + F(n-3)$

Extension 2

Go to the next step using the same technique:

$$2.L(n) = F(n+3) + F(n-3)$$
$$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$$

$$3.L(n) = F(n+4) - F(n-4)$$

[3] The formal proof is let to the reader

Extension 2

Go to the next step using the same technique:

$2.L(n) = F(n+3) + F(n-3)$
$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$

$3.L(n) = F(n+4) - F(n-4)$

One more step: $5.L(n) = F(n+5) + F(n-5)$

[3] The formal proof is let to the reader

Extension 2

Go to the next step using the same technique:

$2.L(n) = F(n+3) + F(n-3)$
$= F(n+4) - F(n+2) + F(n-2) - F(n-4)$

$3.L(n) = F(n+4) - F(n-4)$

One more step: $5.L(n) = F(n+5) + F(n-5)$

Thus, we guess the following expression.

Proposition[3]**.**
$F(k-1).L(n) = F(n+k) + (-1)^{k-1}F(n-k)$ for $k \leq n$

[3] The formal proof is let to the reader

A natural question

The golden ratio.

It is a well-known result that the ratio of two consecutive Fibonacci number tends to the Golden ratio:

- $lim_{n \to \infty} \frac{F(n)}{F(n-1)} = \Phi$

As this result is obtained by solving the following equation $x^2 = x + 1$ (Φ is the positive root) and does not depend on the first rank, this holds also for the Lucas' numbers.

A last result: the Zeckendorf's Theorem

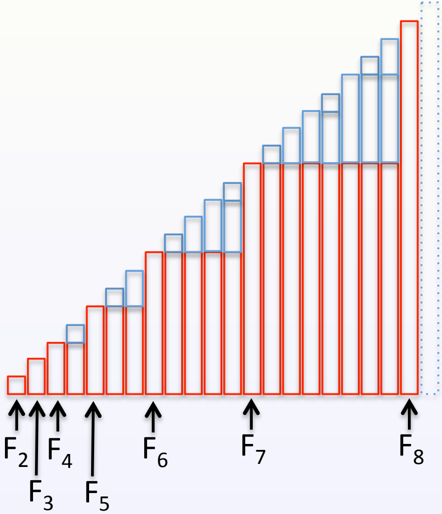
Objective: Study the Fibonacci numbers as a numbering system. Here, we assume that the Fibonacci sequence starts at index 1 and not 0.

Let us first introduce a notation: $j \gg k$ iff $j \geq k + 2$.
The *Zeckendorf's theorem* states that:

every positive integer n has a unique decomposition of the form:
$n = F_{k_1} + F_{k_2} + ... + F_{k_r}$ where $k_1 \gg k_2 \gg ... \gg k_r$ and $k_r \geq 2$

The decompositions will never consider F_1 (since $F_1 = F_2$).

Get intuition of the proof with a picture



Stern's sequence

Definition

$s(0) = 0$ and $s(1) = 1$

$s(2n) = s(n)$
and $s(2n + 1) = s(n) + s(n + 1)$

Stern's sequence

Definition

$s(0) = 0$ and $s(1) = 1$

$s(2n) = s(n)$
and $s(2n + 1) = s(n) + s(n + 1)$

Interpretation:

- If n is even, we keep the value $s(n/2)$
- If it is odd, we split it into two parts that are as balanced as possible.

Cultural aside

- Our purpose in the analysis of Stern (and other) progression is not to study the progression for itself
- but to develop insight about a mathematical object and learn/experience proof techniques

Get a first insight

First elements

1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1 6

Get a first insight

First elements

1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1	6
1	**1**	2	**1**	3	2	3	**1**	4	3	5	2	5	3	4	**1**	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	**1**	6

- What is the *best* representation?

- Consider $s(n)$ as a double entry array:
  $s'(p, q)$, for $p \geq 0$ and $1 \leq q \leq 2^p$

1																							
1	2																						
1	3	2	3																				
1	4	3	5	2	5	3	4																
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

What is the correspondence between elements in s and s'?

1																							
1	2																						
1	3	2	3																				
1	4	3	5	2	5	3	4																
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11

- Consider $s(n)$ as a double entry array:
  $s'(p,q) = s(2^p - 1 + q)$, for $p \geq 0$ and $1 \leq q \leq 2^p$

Progression of the elements along a given column

- It is an arithmetic progression.

1																	
1	2																
1	3	2	3														
1	4	3	5	2	5	3	4										
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5		
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9
1	7	6	11	5	14	9	13	4	15	11	18	7	17	10	13	3	14
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
	+1	**+1**	**+2**	**+1**	**+3**	**+2**	**+3**	**+1**	**+4**	**+3**	**+5**	**+2**	**+5**	**+3**	**+4**	**+1**	**...**

- What is the argument of the corresponding proof?

Proof

Mathematically, we want to prove that $s'(p,q) = s'(p',q) + C_q$ where q is fixed and thus, C_q is a constant.
Let $P(n)$ the following property by induction on n.
$s(2^p + k) = s(k) + s(2^p - k)$

Sum of each row

1 $\longrightarrow$ 1
1 2 $\longrightarrow$ 3
1 3 2 3 $\longrightarrow$ 9
1 4 3 5 2 5 3 4 $\longrightarrow$ 27
1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 $\longrightarrow$ 81

Sum of each row

1	1
1 2	3
1 3 2 3	9
1 4 3 5 2 5 3 4	27
1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5	81

- Successive powers of 3
- Prove this result (a natural way is by induction on p).

Looking carefully at row p

- **p=1**
  $s(2) + s(3) = 2s(2) + s(1) = 3$
- **p=2**
  $s(4) = s(2)$
  $s(5) = s(3) + s(2)$ and $s(3) = s(2) + s(1)$
  $s(6) = s(3) = s(2) + s(1)$
  $s(7) = s(4) + s(3) = 2s(2) + s(1)$

  total of this row: $6s(2) + 3s(1)$ 3 times the previous row.
- **p=3**
  A similar reasoning leads to:
  $18s(2) + 9s(1) = 3[6s(2) + 3s(1)] = 3^2[2s(2) + s(1)]$
- We guess:
  $3^{(p-1)}[2s(2) + s(1)] = 3^p$ that is proven by recurrence

Guess a relation between the terms in a row

1																							
1	2																						
1	3	2	3																				
1	4	3	5	2	5	3	4																
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5								
1	6	5	9	4	11	7	10	3	11	8	13	5	12	7	9	2	9	7	12	5	13	8	11

- They are arranged in a symmetric order and more precisely, like a palindrome.

Proof

- The *pivot* of row p is at $q = 2^{p-1} + 1$
  Thus, it corresponds to $n = 3 \cdot 2^{p-1}$

p=4	1	5	4	7	3	8	5	7	**2**	7	5	8	3	7	4	5

Maximum number in each row

1															
1								**2**							
1				**3**				2				3			
1		4		3		**5**		2		5		3		4	
1	5	4	7	3	**8**	5	7	2	7	5	8	3	7	4	5

[4] again...

Maximum number in each row

1															
1								**2**							
1				**3**				2				3			
1		4		3		**5**		2		5		3		4	
1	5	4	7	3	**8**	5	7	2	7	5	8	3	7	4	5

- They are the successive Fibonacci numbers[4]

[4] again...

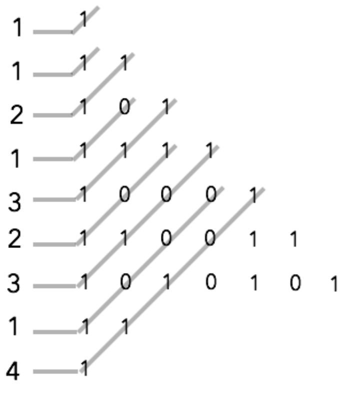
Enumeration of the rationals

- Deriving the rationals

1/1	2/1		3/1				4/1
1/2		2/3				3/4	
	3/2				5/3		
1/3				2/5			
			5/2				
		3/5					
	4/3						
1/4							

Link with the Pascal's triangle

Consider the Triangle modulo 2.

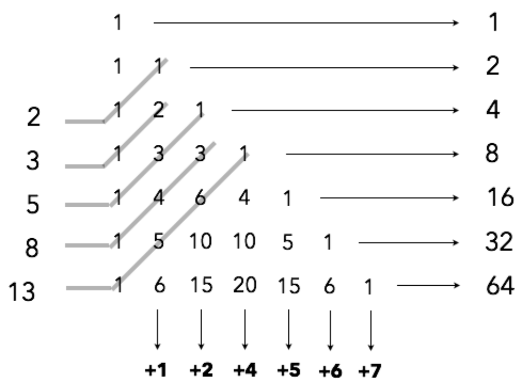


- Combinatorial interpretation of $s(n)$:
  $\sum_{i,j,2i+j=n} \binom{i+j}{i}$ modulo 2.

Similarities between the two sequences

- The rows in Pascal triangle sum up to powers (of 2)
- arithmetic progression along columns
- Deriving Fibonacci numbers
- Symmetry within the rows

In a picture


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
2
3
5
8
13
1
2
4
8
16
32
64
+1 +2 +4 +5 +6 +7