Maths for Computer Science
Basic on graphs
Part. 2: Structured Graphs (end)

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Matching

Definition

A matching is a set of edges that have no vertices in common.

It is *perfect* if its vertices are all belonging to an edge[1].

Proposition.

The number of perfect matchings in a graph of order $n = 2k$ grows exponentially with k.

[1] thus, the number of vertices is even and the cardinality of the matching is exactly half of this number

Example

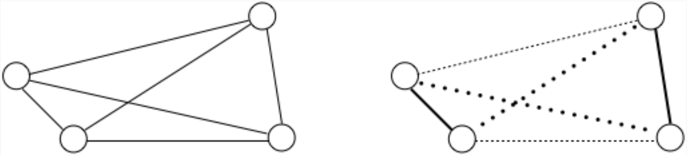

Figure: The 3 possible perfect matchings of K_4.

Proof

by recurrence on k,
let denote the number of perfect matchings by N_k.

Base case: For $k = 1$, there is only one perfect matching $N_1 = 1$ and for $k = 2$, there are 3 different perfect matchings $N_2 = 3$.
Induction step: For k, there are $2k - 1$ possibilities for a vertex to choose an edge, $N_k = (2k - 1).N_{k-1}$
Thus, N_k is the product of the k first odd numbers.

However, determining a perfect matching of minimal weight in a weighted graph can be obtained in polynomial time (using the Hungarian assignment algorithm).

Another interesting class of graphs.

Bipartite graphs.

A graph G is bipartite if its vertices can be partitioned into (by definition of partition, disjoint) sets X and Y in such a way that every edge of G has one endpoint in X and the other in Y.

An interesting question is to link bipartite graphs and matchings.

Trees

Definition

Trees are identified mathematically as graphs that contain no cycles or, equivalently, as graphs in which each pair of vertices is connected by a unique path.

A tree is thus the embodiment of “pure” connectivity, which provides the minimal interconnection structure (in number of edges) that provides paths that connect every pair of vertices.

Proposition

Any tree of order n has $n-1$ edges.

Example

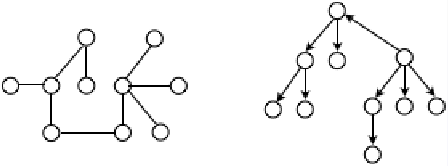

Figure: Undirected and directed trees.

Preliminary results

Lemma

1. Let G be a connected graph with $n \geq 2$ vertices.
   Every vertex of G has degree at least 1.
2. Any connected tree of order n ($n \geq 1$) has at least one vertex of degree 1 (called a leaf).

Preliminary results

Lemma

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Rapid Proofs

The main argument is on the analysis of graphs with minimum degrees 0 for part 1 and more than 2 for part 2.

Proof 1

Principle

By induction on the order of the graph n.

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By induction on the order of the graph n.

Base case for $n = 2$
Induction step. Use the previous Lemma.

Proof 2

Inductive hypothesis. Assume that the indicated tally is correct for all trees having no more than k vertices.

Inductive extension. Consider a tree T with $k+1$ vertices.

By the Lemma, T must contain at least one vertex v of degree 1.
If we remove v and its (single) incident edge, we now have a tree T' on k vertices.
By induction, T' has $k-1$ edges. When we reattach vertex v to T', we restore T to its original state.
Because this restoration adds one vertex and one edge to T', T has $k+1$ vertices and k edges.

Spanning Trees

Let consider a weighted graph G.

Motivation

a way of succinctly summarizing the connectivity structure inherent in undirected graphs.

Definition

Take the same set of vertices and extract a set of edges that spans the vertices.

Determining a minimal Spanning Tree is a polynomial problem. There exist two possible constructions, following two different philosophies.