Master of Science in Informatics at Grenoble

Grenoble INP Ensimag

UE Mathematics for Computer Science

Final exam December 14, 2010 (3 hours)
Neither documents nor calculators are allowed. Only personal hand-written notes are allowed. Use separated sheets for problems 1-2 and problems 3-4-5. All problems are independent from each other.

Problem 1: Cake division *(6 points)*

This problem investigates some strategies to divide a "cake" fairly. We are interested here in dividing some resources fairly, which means that all recipients believe that they have received a fair amount of resources. Each recipient has a different "measure" of the value of the pieces of the resources. For instance, in the cake version of the problem, one recipient may like marzipan and another one would rather prefer chocolate... There exist many variants of the problem. The meaning of fair may simply be a proportional sharing of resources, or harder requirements like "envy-free" may also need to be satisfied. Proportional fair division guarantees each recipient (player) obtains a fair share. For instance, if three people divide up a cake each gets at least a third by their own valuation. An envy-free division guarantees no one will prefer somebody else's share more than their own. The assumptions about the valuation of the resources (pieces of cake) are: Each player has their own opinion of the value of each part of the resources. The value (also called measure) of a player of any allocation is the sum of his/her valuations of each part. This corresponds to the additive property of the measure. The measure of a player is not known by the others. The resources can be divided into parts with arbitrarily small value.

A fair division protocol lists the actions to be performed by the players in terms of the visible data and their valuations. A valid procedure is one that guarantees a fair division for every player who acts rationally according to their valuation. Where an action depends on a player's valuation the procedure is describing the strategy a rational player will follow. A player may act as if a piece had a different value but must be consistent. For instance if a procedure says the first player cuts the cake in two equal parts then the second player chooses a piece, then the first player cannot claim that the second player got more.
What the players do is:
-Agree on their criteria for a fair division;
-Select a valid procedure and strictly follow its rules.
Finally, we assume that the objective of each player is to maximize the minimum amount they might obtain.

Question 1.1 :

For two recipients, there is a simple solution which is commonly employed. This is the so-called "divide-and-choose" method. One person divides the resource into what they believe are equal halves, and the other one chooses the ”half” he/she prefers. Give briefly the argument why the person making the division has an incentive to divide as fairly as possible. Show that this strategy provides an envy-free division. Construct a simple example that shows that this procedure is not equitable.

Question 1.2 :

The moving-knife procedure gives an exact division for two players. First, let us assume that there exists an external referee who is managing the knife. He places the knife over the cake at the left side and move it slowly to the right. Each of both payers can stop the knife at any moment. Then, the first player who stops the knife obtain the correspond left piece of the cake, the other gets the right one. Show that there exists a moment when the division is fair.

Question 1.3 :

Now, we want to remove the external referee. The first player places two knives over the cake such that one knife is at the left side of the cake and one is further right; half of the cake lies between the knives. He then moves the knives right, always ensuring there is half the cake – by his valuation – between the knives. If he reaches the right side of the cake, the leftmost knife must be where the rightmost knife started off. The second player stops the knives when he thinks there is half the cake between the knives (according to his own evaluation).

Give the argument why there is always a point at which this happens.

This moving knife strategy can be extended to more than two payers (let say three). Show with an example that the first player to stop will produce an envy-free division.

Problem 2: Flooding *(3 or 4 points)*

Let us consider three bottles that contains each an integer number of water. Each one is large enough to contain the whole quantity of water. The only allowed operation is to double the content of a bottle by pouring water from another bottle.

Question 2.1 :

Show that it is always possible to empty one of the bottle, whatever is the initial configuration.

Problem 3: Boolean expressions and probability *(6 points)*

Here are seven propositions on nine boolean variables:

$$x_1 \vee x_3 \vee \neg x_7, \quad \neg x_5 \vee x_6 \vee x_7, \quad x_2 \vee \neg x_4 \vee x_6, \quad \neg x_4 \vee x_5 \vee x_7,$$

$$x_3 \vee \neg x_5 \vee \neg x_8, \quad x_9 \vee \neg x_8 \vee x_2, \quad \neg x_3 \vee x_9 \vee \neg x_4$$

Note that:

1. Each proposition is the OR of three terms of the form x_i or the form $\neg x_i$ (not x_i).
2. The variables in the three terms in each proposition are all different.

Question 3.1 : Counting

How many different propositions of three terms could be written with nine variables.

Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

Question 3.2 : Modeling

Propose a probabilistic modeling of this problem and evaluate the probability that the first proposition is true.

Question 3.3 : Number of true propositions

What is the expected number of true propositions?

Question 3.4 :

Use your answer to prove that there exists an assignment to the variables that makes all of the propositions true.

Problem 4: Mother's Day *(6 points)*

You would like to give a bouquet for Mother's Day, but you know nothing about flower arrangement. You google and find an online service where you just enter the number of each type of flower you want, and they make a nice bouquet and send it to your Mother. You decide to buy a bouquet of lilies, roses and tulips, but with the following restrictions:
- there must be at most 3 lilies,
- there must be an odd number of tulips.
You don't specify any restriction on the number of roses, leaving the online service with some flexibility in making up the bouquet. For example, a bouquet of 3 tulips, 5 roses and no lilies satisfies the restrictions.

Suppose the service decides to assemble a bouquet of n flowers, and let f_n be the number of possibilities they have to assemble the bouquet satisfying your restrictions.

Question 4.1 : Small bouquet

Compute f_0, f_1,..., f_5

Denote by n_1 (resp. n_2 and n_3) the number respectively of lilies, roses and tulips.

Question 4.2 : Integer equation

Write the constraints on the n_i and express f_n as the number of solutions of an inequality with integer variables.

Let $F(x)$ be the generating function for the sequence $f_0, f_1, f_2, ...$.

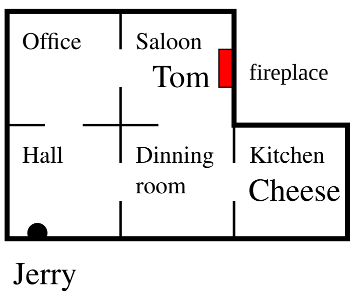
Question 4.3 : Closed formula

Express F(x) as a quotient of polynomials (or products of polynomials).
Give a closed formula for f_n (Bonus question).

Tom and Jerry *(4 points)*

Tom and Jerry have been left alone at home. Jerry is hungry and begins to search some piece of cheese that had been left somewhere in the house (in fact in the kitchen). If Jerry enters the kitchen, he immediately runs inside the big piece of gruyere and as a glutton fills his stomach. Usually Tom is tired and sleeps in front of the fire-place in the saloon. But today Tom is awake and hunts the poor Jerry from room to room (starting from the fireplace). At each time step either Tom or Jerry moves into another room with probability $\frac{1}{2}$. When Tom and Jerry meet or Jerry is in the gruyere the game is over.

Home sweet home



Question 5.1 : Modelling

Propose a Markov model of this system, the state is modeled by the rooms where Tom and Jerry stand. Draw the state/transition graph and give the transition matrix. Classify the states of this Markov chain.

Question 5.2 : The end of the game

Propose a method to compute the probability that Jerry meets the cheese before Tom. Compute this probability (bonus).