
UE Mathematics for Computer Science

Final exam December 19, 2012 (3 hours)
Only personal hand-written notes are allowed.
Use separated sheets for problems 1-2 (part I) and problems 3-4 (part II).
All problems are independent from each other.
Number of points given for each problem is given for information purposes only and is subject to modifications without notice.

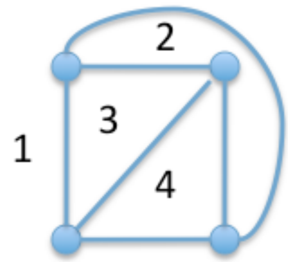
Part I

Problem 1: Planar Graphs *(6 points)*

Question 1.1 : Preliminary example

> Let consider three dogs and three neighboring houses, can you find a path from each dog to each house such that no two paths intersect ?

The drawing below has four faces:



Face 1, which extends off to infinity in all directions, is called the outside face. It turns out that the number of vertices and edges in a connected planar graph determine the number of faces in every drawing. This result is known as Euler formula, we are going to prove it.

> **For every drawing of a connected planar graph $\mathcal{G}$ the following expression holds:**
>
> $n - m + f = 2$
>
> where n is the number of vertices, m is the number of edges, and f is the number of faces.

Let remark that this formula is true for graph above, $n = 4$, $m = 6$, and $f = 4$.

The proof is by induction on the number of edges. Let $\mathcal{P}(m)$ be the proposition that $n - m + f = 2$ for every drawing of a planar graph $\mathcal{G}$ with m edges.
Question 1.2 :

> Prove the basis of the induction.

Consider now a connected planar graph $\mathcal{G}$ with $m + 1$ edges.
Question 1.3 :

Prove the induction for acyclic graphs.

Question 1.4 :

We consider now a graph with (at least) a cycle (let denote it by $\mathcal{C}$). Select a spanning tree and an edge (u, v) in $\mathcal{C}$, but not in the tree. Sow how to apply the induction hypothesis by removing (u, v).

Problem 3: Tournaments *(4 points)*

Suppose that n players compete in a round-robin tournament. Thus, for every pair of players u and v, either u beats v or else v beats u. Interpreting the results of a round-robin tournament can be problematic. There might be all sorts of cycles where x beat y, y beat z, yet z beat x. Graph theory provides at least a partial solution to this problem.

The results of a round-robin tournament can be represented with a tournament graph. This is a directed graph in which the vertices represent players and the edges indicate the outcomes of games. In particular, an edge from u to v indicates that player u defeated player v. In a round-robin tournament, every pair of players has a match. Thus, in a tournament graph there is either an edge from u to v or an edge from v to u for every pair of vertices u and v. Recall that a directed Hamiltonian path is a directed walk that visits every vertex exactly once.
Question 2.1 :

Show that in every round-robin tournament, there exists a ranking of the players such that each player lost to the player ranked one position higher. In other words, every tournament graph contains a directed Hamiltonian path.

Hint: The proof is by induction. Let $\mathcal{P}(n)$ be the proposition that every tournament graph with n vertices contains a directed Hamiltonian path. The idea is to isolate an arbitrary vertex and to construct a hamiltonian path using paths of lower dimensions.

Part II

Problem 3: Labels *(5 points)*

A label identifier, for a computer system, consists of one letter followed by three digits.
Question 3.1 : No constrained labels

If repetitions are allowed, how many distinct label identifiers are possible ?

To check the validity of the label, we assume that the sum of the digits are 0 $modulo$ 3.
Question 3.2 : No constrained labels

How many distinct label identifiers, with checksum 0 modulo 3 are possible ?

Question 3.3 : Label generator

Design an algorithm that generates uniformly a random label and prove that it is a uniform generator. We suppose given a `random()` function that provides a sequence of independent real numbers uniformly distributed on $[0, 1)$

Problem 4: Monotonicity *(5 points)*

Question 4.1 :

Compute $f(m, n)$ the number of functions from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$.

Question 4.2 :

Propose a simple algorithm that generates uniformly a function from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$.

Question 4.3 :

Compute the expected number of fixed points of a uniformly generated function from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, m\}$.

A function f is said to be **strictly increasing** if for all $x < y$ we have $f(x) < f(y)$.

Question 4.4 :

For $m \leqslant n$ use combinatorial arguments to compute $c(m, n)$ the number of strictly increasing functions from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$.

A function f is said to be **nondecreasing** if for all $x < y$ we have $f(x) \leqslant f(y)$.

Question 4.5 :

Use combinatorial arguments to compute $d(m, n)$ the number of nondecreasing functions from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$.

Question 4.6 :

Design an algorithm that generates uniformly a nondecreasing function from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$.

Question 4.7 : (bonus)

Compute the expected number of fixed points of a uniformly generated nondecreasing function from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, m\}$.