Fundamental Computer Science
Training session NP-completeness

Denis Trystram
MoSIG1 and M1Info – University Grenoble-Alpes

March, 2021

Agenda

- Horn-SAT
- 2SAT
- analysis of CLIQUE
- Dynamic Programming for SubSetSum
- Bin Packing

Complexity of Horn-SAT

A Horn formula has at most one positive literal per clause.

$$\textsc{Horn}\text{-SAT} = \{\langle \mathcal{F} \rangle \mid \mathcal{F} \text{ is a satisfiable Horn formula}\}$$

Recall:

- Positive literal: x_i
- Negative literal: $\bar{x}_i$

Complexity of Horn-SAT

A Horn formula has at most one positive literal per clause.

$$\textsc{Horn}\text{-SAT} = \{\langle \mathcal{F} \rangle \mid \mathcal{F} \text{ is a satisfiable Horn formula}\}$$

Recall:

- Positive literal: x_i
- Negative literal: $\bar{x}_i$

Prove that $\textsc{Horn}\text{-SAT} \in \mathcal{P}$

Tipp:

- What has to happen to clauses that contain only one single literal?
- Consider the case that each clause contains a negative literal.

Solution Horn-SAT

Algorithm

1. **While** there are clauses with only one literal
   - pic a clause with only one literal
   - set the corresponding variable to T or F such that the clause is satisfied
   - delete all the other clauses that are satisfied by this assignment and remove the variable from all the other clauses
2. set all non-assigned variables to F

Solution Horn-SAT

Algorithm

1. **While** there are clauses with only one literal
   - pic a clause with only one literal
   - set the corresponding variable to T or F such that the clause is satisfied
   - delete all the other clauses that are satisfied by this assignment and remove the variable from all the other clauses
2. set all non-assigned variables to F

Sketch of the analysis:
After step 1 all the clauses contain at least one negative literal.
Therefore, after setting all variables to F in step 2, every clause will contain at least one literal that is T.
Hence, all the clauses are satisfied.

Complexity is in $O((n \cdot m)^2)$

2SAT

- $X = \{x_1, x_2, \ldots, x_n\}$: set of variables
- $C = \{C_1, C_2, \ldots, C_m\}$: set of clauses fo cardinality 2
- $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

$$\text{SAT} = \{\langle \mathcal{F} \rangle \mid \mathcal{F} \text{ is a satisfiable Boolean formula } \}$$

Prove $2\text{SAT} \in \mathcal{P}$

The solution is detailed in the slides of lecture 4: *variants of SAT*.

Presentation of CLIQUE

$\text{CLIQUE} = \{\langle G, k\rangle \mid G = (V, E)$ is a graph with a subset of vertices A of cardinality k and for each pair of vertices in A, $(x, y) \in E$ $\}$

CLIQUE $\in$ NP-COMPLETE

CLIQUE $\in \mathcal{NP}$

- given a set of vertices, check if there is an edge between any pair of them

CLIQUE $\in$ NP-complete

CLIQUE $\in \mathcal{NP}$

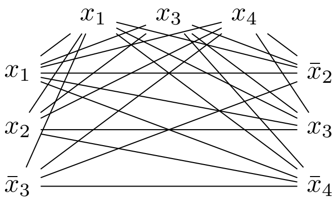
- given a set of vertices, check if there is an edge between any pair of them

3SAT $\leq_{\mathrm{P}}$ CLIQUE

1. given any formula $\mathcal{F}$ of SAT, we construct an instance $I = \langle G, k \rangle$ of CLIQUE
   - add a vertex for each literal
   - add an edge between any two literals except:
     (a) literals in the same clause
     (b) a literal and its negation
   - $k = m$ (number of clauses)

Example

$$\mathcal{F} = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4)$$



CLIQUE $\in$ NP-COMPLETE

3SAT $\leq_{\text{P}}$ CLIQUE

2. $|V| = 3m$, $|E| = O(m^2)$

CLIQUE $\in$ NP-COMPLETE

3SAT $\leq_P$ CLIQUE

2. $|V| = 3m$, $|E| = O(m^2)$

3. $\mathcal{F}$ is satisfiable iff there is a clique of size k in G
   - assume that $\mathcal{F}$ is satisfiable
   - at least one literal is TRUE in any clause
   - there is an edge between such literals (why?)
   - hence, the corresponding vertices form a k-clique

CLIQUE $\in$ NP-COMPLETE

3SAT $\leq_{\text{P}}$ CLIQUE

2. $|V| = 3m$, $|E| = O(m^2)$
3. $\mathcal{F}$ is satisfiable iff there is a clique of size k in G
   - assume that $\mathcal{F}$ is satisfiable
   - at least one literal is TRUE in any clause
   - there is an edge between such literals (why?)
   - hence, the corresponding vertices form a k-clique

   - assume there is a k-clique in G
   - this clique contains at most one vertex from each clause
   - $k = m$, hence the clique contains exactly one vertex from each clause
   - each pair of these vertices is compatible (no a literal and its negation)
   - set the corresponding literals to TRUE
   - $\mathcal{F}$ is satisfiable

Solving SubsetSum

SubsetSum

Input: a set of positive integers $A = \{a_1, a_2, \ldots, a_k\}$
$t \in \mathbb{N}$

Question: is there a set $B \subseteq A$ such that $\sum_{a_i \in B} a_i = t$?

Write a dynamic programming algorithm for solving this problem.

Solving SubsetSum

SubsetSum

Input: a set of positive integers $A = \{a_1, a_2, \ldots, a_k\}$
$t \in \mathbb{N}$

Question: is there a set $B \subseteq A$ such that $\sum_{a_i \in B} a_i = t$?

Write a dynamic programming algorithm for solving this problem.

Tip:

- Consider the integers sorted in non-decreasing order: $a_1 \leq a_2 \leq \ldots \leq a_n$
- $S[i, q] = \begin{cases} \text{True}, & \text{if there is a SubsetSum among the } i \text{ first integers which sums up exactly to } q \\ \text{False}, & \text{otherwise} \end{cases}$

The detailed solution is in the slides of Lecture 4 *pseudo-polynomial algorithms*.

Bin Packing

Bin-Packing

Input: a set of items A, a size $s(a)$ for each $a \in A$, a positive integer capacity C, and a positive integer k

Question: is there a partition of A into disjoint sets $A_1, A_2, \ldots, A_k$ such that the total size of the elements in each set A_j does not exceed the capacity C, i.e., $\sum_{a \in A_j} s(a) \leq C$?

Show that this problem is NP-complete
Is it strongly or weakly NP-complete?
(try to give the strongest result)