Maths for Computer Science
Multiple ways for solving a problem

Denis TRYSTRAM
Lecture notes MoSIG1

sept. 8, 2022

Brief recall on Triangular numbers

Definition:

Triangular numbers are defined as the sum of the n first integers:
$\Delta_n = \sum_{k=1}^{n} k$

Brief recall on Triangular numbers

Definition:
Triangular numbers are defined as the sum of the n first integers:
$\Delta_n = \sum_{k=1}^{n} k$

- There exist many proofs for establishing the value of Δ_n
- The most interesting ones are studied below

The Gauss trick

- One of the most popular technique is obtained in writing the sum forward and backward and gathering the terms two by two as follows:

$$
\begin{array}{rcccccccc}
2.\Delta_n & = & 1 & + & 2 & + & \ldots & + & n \\
 & + & n & + & n-1 & + & \ldots & + & 1
\end{array}
$$

$$= (n+1) + (n+1) + \ldots + (n+1)$$

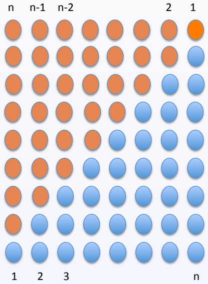
$2\Delta_n$ is equal to n times $n+1$, thus, $\Delta_n = \frac{n \cdot (n+1)}{2}$

Another way of looking at this process

- The previous proof comes from Arithmetic manipulations
- Let us use the double counting Fubini principle.
  It may be a way to gain insight of using the trick...

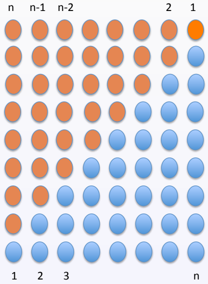
Double counting

Δ_n is represented by piles of tokens arranged as a triangle.
Putting two copies up side down gives the following n by $n+1$ rectangle.



Double counting

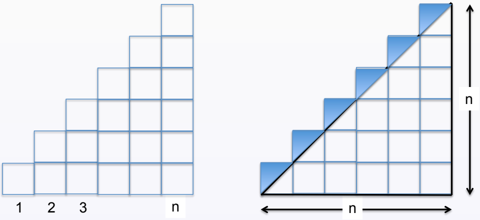
Δ_n is represented by piles of tokens arranged as a triangle. Putting two copies up side down gives the following n by $n+1$ rectangle.



Notice here that if we replace the tokens by the number 1, the problem becomes arithmetic...

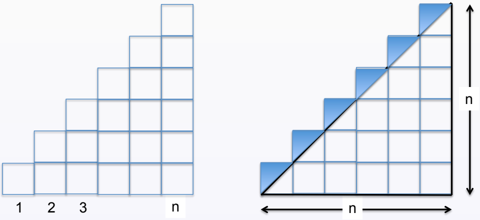
Another way of looking at this process

The following figure proves the same result by using a geometric argument instead of tokens.



Another way of looking at this process

The following figure proves the same result by using a geometric argument instead of tokens.



The sum is represented by a tesselation of squares of size 1 by 1.
The global is obtained by the surface of half the *big* square ($\frac{n^2}{2}$)
plus n times half of the surface of the unit diagonal squares
Thus, $\frac{n^2}{2} + n.\frac{1}{2} = \frac{(n+1).n}{2}$

Combinatorial proof

- Sometimes combinatorial argumentation can be used in unexpected ways.
- We illustrate this by deriving an explicit expression for the summation as a selection problem

  $$S_{n-1} \ = \ 1 \ + \ 2 \ + \cdots + \ (n-1)$$

Our summation starts by counting the number of ways of selecting two items from a set of n items
call it $C(n, 2)$

- The first integer of the two we are selecting can be chosen in $n-1$ ways, corresponding to the $n-1$ elements of the set
$$\{1,\ 2, \ldots,\ (n-2),\ (n-1)\}$$
- If the bigger integer chosen was k, then we can select the second, smaller integer in $k-1$ ways, from among the integers smaller than k.

We thereby observe the following summation:

$$\begin{aligned} C(n,2) &= (n-1) + \sum_{k=2}^{n-1} (k-1) \\ &= (n-1) + \sum_{k=1}^{n-2} k \\ &= S_{n-1} \end{aligned}$$

Probabilist proof

- The most elegant and simple proof is based on a probabilistic argument.
- What is the average of the first 100 integers?

Probabilist proof

- The most elegant and simple proof is based on a probabilistic argument.
- What is the average of the first 100 integers?
- We deduce immediately the sum, equal to 100 times the average.
- This can obviously be extended for any n.

Concluding remarks

We presented in here several ways for solving the same problem.

Take home message:

- The study of various methods gave more insight of the triangular numbers them selves
- There are many links from a method the another
- Everyone can find her/his own method!