Fundamental Computer Science
Training on some NP-complete problems

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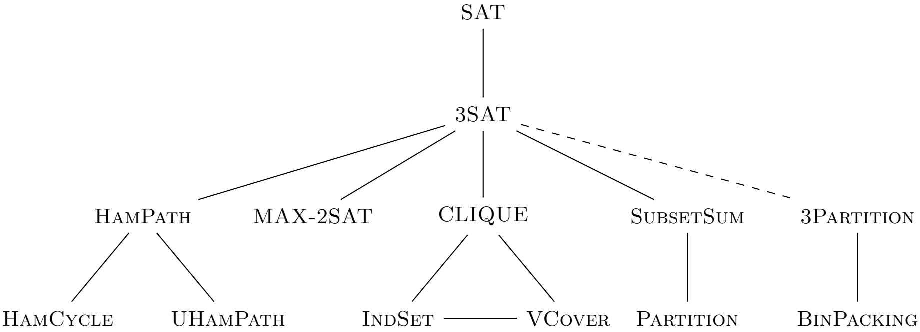
Some NP-complete problems



Today

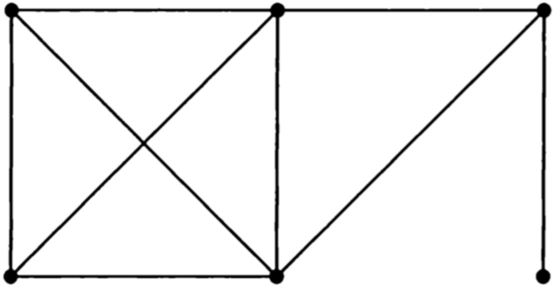
- vertex cover
- 2-Partition
- Knapsack

Vertex cover

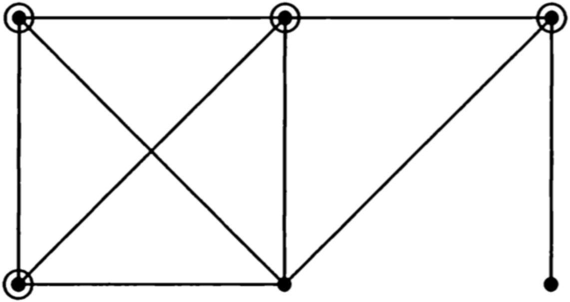
Input: a graph $G = (V, E)$ and an integer j.

Question: is there a subset of the vertices whose cardinality is less than j that covers all the edges of G?

Example 1: initial graph



Example 2: a set cover



VC belongs to NP

- Generate non deterministically a set of vertices.
- Verify that these vertices cover all the edges by a polynomial-time algorithm.

Reduction

We will show that 3SAT α VC.

Let consider an instance of 3SAT:
$E_1 \wedge E_2 ... \wedge E_k$ (1)
where $E_i = x_{i,1} \vee x_{i,2} \vee x_{i,3}$

Let denote by p_q (for $q = 1$ to l) the set of propositional variables in (1)

Construction of the instance of VC

- **Set of vertices**
- A pair of vertices $(p_q, \bar{p}_q)$ for each of the propositional variable
- A triple of vertices associated to each clause E_i

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Number of vertices:
$2l + 3k$

Construction of the instance of VC

- **Set of edges**
- An edge between each pair of vertices $(p_q, \bar{p}_q)$
- An edge between each pair $(x_{i,1}, x_{i,2})$, $(x_{i,1}, x_{i,3})$ and $(x_{i,2}, x_{i,3})$
- one edge between each $x_{i,j}$ and p or $\bar{p}$ depending of the literal

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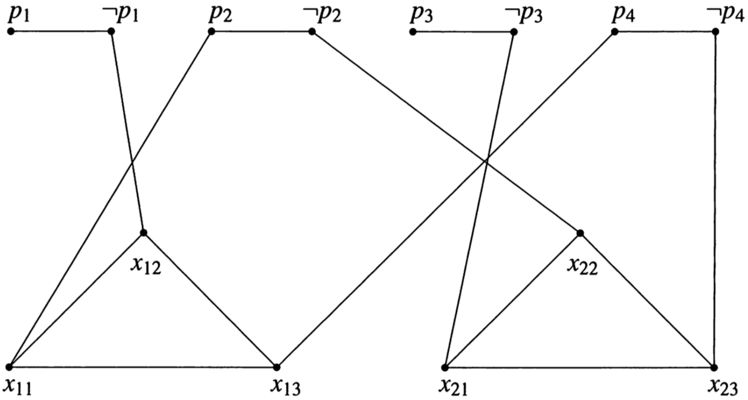
the constant $j = l + 2k$

Example

$(p_2 \vee \bar{p}_1 \vee p_4) \wedge (\bar{p}_3 \vee \bar{p}_2 \vee \bar{p}_4)$

Draw the graph.

Example: the corresponding graph



Reduction

Write the detailed reduction.

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Goal

show that the instance of 3SAT is satisfiable iff the graph generated by the reduction is a vertex covering.

Reduction 2

Let assume that the boolean expression is satisfiable.

This means that all the clauses are true.
The vertex cover is defined by this interpretation function:

- the vertices p_q if the interpretation function is equal to 1 and $\bar{p}_q$ otherwise.
- two vertices among the three into a triangle, such that the interpretation function leads to 1 for the not chosen vertex[1]

The size of this covering is $l + 2k$.

We verify easily that it covers all the edges.

[1] it is always possible

Reduction 3

Assume now that the graph has a covering of size $l + 2k$.
We have to show that the corresponding boolean expression (3SAT) is satisfiable.

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- such a covering should contain at least one vertex among the pair p_i and $\bar{p}_i$
- it should also contain two vertices among $x_{i,1}$, $x_{i,2}$ and $x_{i,3}$ in order to cover the triangles
- it can not contain other vertices

2Partition

Instance : n integers denoted by n_i and an integer (even) $S = \Sigma_{1 \leq i \leq n} n_i$.
Question : Does it exist a partition of these integers into two subsets A_1 et A_2 such that $\Sigma_{i \in A_1} n_i = \Sigma_{i \in A_2} n_i$?