Bi-objective Scheduling Algorithms for Optimizing Makespan and Reliability on Heterogeneous Systems

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Outline of the talk

Introduction

Problem studied:

- scheduling DAG
- heterogeneous systems
- hardware can fail

Bi-criteria objective:

- given a makespan objective
- optimize reliability

Related work

A "*new subject*" :

- Dogan & Ozgüner 2002: Model the problem, RDLS bi-criteria heuristic.
- Dogan & Ozgüner 2004: enhancement of previous result (GA).
- Qin & Jiang 2005: first optimize deadline, then maximize reliability.
- Hakem & Butelle 2006: BSA, bi-criteria heuristic that outperforms RDLS.

Modeling

- $G = (V, E)$: a DAG.
- $v_i \in V$ is associated a number of operations: o_i.
- $n = |V|$
- $e_i = (i, j) \in E$ is associated l_i the time to send data from task v_i to task v_j (if they are not executed on the processor).
- a set P of m processors
- processor $p_j \in P$ is associated with two values:
  - τ_j the time to perform one operation and
  - λ_j the failure rate.
- v_i executed on p_j will last $o_i \times \tau_j$.

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Assumption:

- During the execution of the DAG, the failure rate is constant.

$\Rightarrow$ failure model follows an exponential law.

$\Rightarrow$ probability that v_i finishes (correctly) its execution:

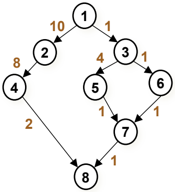
$$e^{-o_i \times \tau_j \times \lambda_j}$$

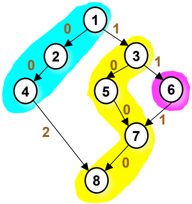
Outline

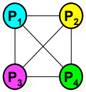
Scheduling problem

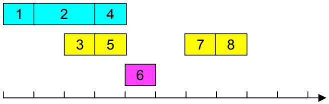
Allocate tasks to processors such that:

- two tasks cannot be allocated to the same processor at the same time,
- dependencies are respected.









Criteria

C_j: termination date of processor j

Two criteria to optimize:

- **Makespan**: minimize

$$M = \max(C_j)$$

- **Reliability**: maximize

$$p_{\text{succ}} = \prod_{j=1}^{m} e^{-C_j \lambda_j} = e^{-\sum_{j=1}^{m} C_j \lambda_j}$$

  or minimize

$$\sum_{j=1}^{m} C_j \lambda_j$$

Two unrelated criteria

Proposition

Let S be a schedule where all the tasks have been assigned, in topological order, to the processor i such that $\lambda_i \tau_i$ **is minimum**. *Then any schedule S' is such that $p'_{succ} \leq p_{succ}$.*

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Proof

- s.w.l.o.g $i = 1$ ($i.$ $e.$, $\forall j:\ \tau_1\lambda_1 \leq \tau_j\lambda_j$).
- $p_{\text{succ}} = e^{-C_1\lambda_1}$, $p'_{\text{succ}} = e^{-\sum_{j=0}^{m} C'_j\lambda_j}$.
- $T = T_2 \cup \ldots \cup T_m$, sets of the tasks allocated to processors $2, \ldots, m$ by S'.
- $C'_1 \geq C_1 - \tau_1 \sum_{v_i \in T} o_i$.
- $\forall\, 2 \leq j \leq m,\ C'_j \geq \tau_j \sum_{v_i \in T_j} o_i$

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$$\sum_{j=1}^{m} C'_j\lambda_j - C_1\lambda_1 \quad \geq \quad \sum_{j=2}^{m} \left((\tau_j\lambda_j - \tau_1\lambda_1) \sum_{v_i \in T_j} o_i \right) \geq 0$$

$$\Rightarrow \frac{p_{\text{succ}}}{p'_{\text{succ}}} = e^{\sum_{j=1}^{m} C'_j\lambda_j - C_1\lambda_1} \geq 1$$

Bi-criteria scheduling

Objective: maximizing the reliability subject to the condition that the makespan is minimized.

- Finding the optimal makespan, is most of the time NP-hard,
- we aim at designing an $(\alpha,\ \beta)$-approximation algorithm.
- $(\alpha,\ \beta)$-approximation algorithm:
  - makespan at most α times larger than the optimal one,
  - probability of failure is at most β times larger than the optimal one (among the schedules that minimize the makespan).

Approximation algorithm and probability

Let p_{succ} (resp. p_{fail}) be the probability of success (resp. of failure) of a schedule S.

Let $\tilde{p}_{\text{succ}}$ (resp. $\tilde{p}_{\text{fail}}$) be the optimal probability of success (resp. of failure) for the same input as S.

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Proof The proof is based on the Taylor's series of $(1-x)^n$, where, $\forall x \in [0,1], \forall n \in [1, +\infty[, (1-x)^n \leq 1 - nx$.

$$\begin{aligned} p_{\text{fail}} &= 1 - p_{\text{succ}} \leq 1 - \tilde{p}_{\text{succ}}^{\beta} = 1 - (1 - \tilde{p}_{\text{fail}})^{\beta} \\ &\leq 1 - (1 - \beta \cdot \tilde{p}_{\text{fail}}) = \beta \cdot \tilde{p}_{\text{fail}} \end{aligned}$$

Outline

Independent unitary tasks

$o_i = 1$ and $E = \emptyset$, $n = |V|$.

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Algorithm 1 Makespan-optimal allocation for independent unitary tasks

for i=1 to P
  $n_i \leftarrow \left\lfloor \frac{1/\tau_i}{\sum 1/\tau_i} \right\rfloor \times n$
while $\sum n_i < n$
  $k = \text{argmin}(\tau_k(n_k + 1))$
  $n_k \leftarrow n_k + 1$

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Above algorithm gives M_{opt} the best achievable makespan.
For the reliability criteria the user gives the value of α that tells how far from the optimal makespan he/she can tolerate to be.
Then we compute a schedule such that:

- $M \leq \alpha M_{\text{opt}}$
- it has the best reliability among all the schedules with makespan $\leq M$.

Optimal algorithm for Independent unitary tasks

Algorithm 2 Optimal reliable allocation for independent unitary tasks

Input: $\alpha \in [1, +\infty[$

Compute $M = \alpha M_{\text{opt}}$ using previous algorithm

Sort the processor by increasing $\lambda_i \tau_i$

$X \leftarrow 0$

for i=1 to P

  if $X < N$

    $n_i \leftarrow \min\left(N - X, \left\lfloor \frac{M}{\tau_i} \right\rfloor\right)$

  else

    $n_i \leftarrow 0$

  $X \leftarrow X + n_i$

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- w.l.o.g. let $n'_1 = n_1 - k$, $n'_i = n_i + k$ and $n'_j = n_j$ for $k \in [1, n_i]$, $j \neq 1$ and $j \neq i$.

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- Then the difference between the two objective values is:

$$\begin{aligned}
X &= n_1\lambda_1\tau_1 + \ldots + n_i\lambda_i\tau_i + \ldots + n_N\lambda_N\tau_N - n'_1\lambda_1\tau_1 - \ldots - n'_i\lambda_i\tau_i - \ldots + n'_N\lambda_N\tau_N \\
&= \lambda_1\tau_1(n_1 - n'_1) + \lambda_i\tau_i(n_i - n'_i) \\
&= k\lambda_1\tau_1 - k\lambda_i\tau_i \\
&= k(\lambda_1\tau_1 - \lambda_i\tau_i) \\
&\leq 0 \text{ because } \lambda_i\tau_i \geq \lambda_1\tau_1.
\end{aligned}$$

Hence, the first allocation has a smaller objective value.

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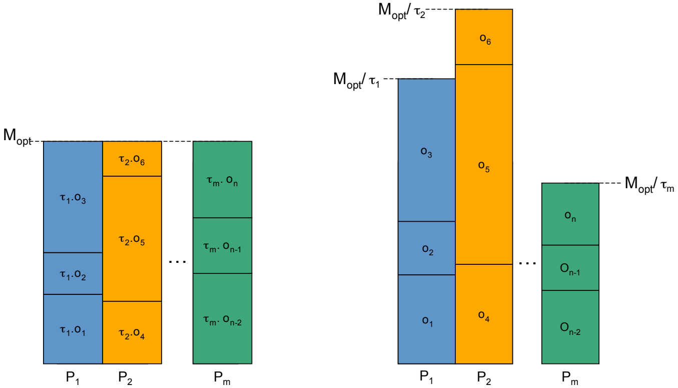

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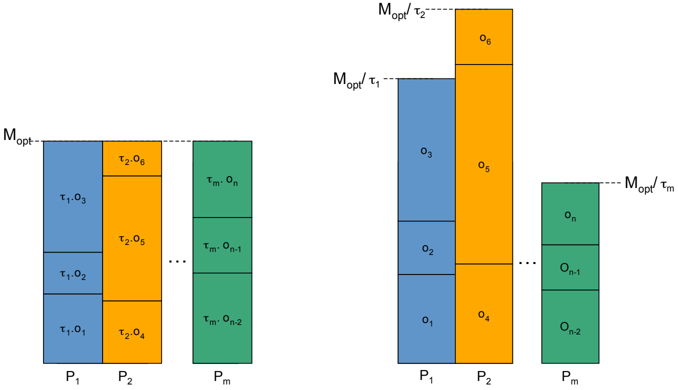

$$\sum_{j=1}^{m} \frac{M_{\text{opt}}}{\tau_i} = \sum_{i=1}^{n} o_i \Rightarrow M_{\text{opt}} = \frac{\sum_{i=1}^{n} o_i}{\sum_{j=1}^{m} \frac{1}{\tau_i}}$$

Independent tasks: the makespan problem

Gonzalez, Ibarra, Sahni 1977:

$$\left\{ \begin{array}{l} M_{\text{opt}} \geq \frac{\sum_{i=1}^{n} o_i}{\sum_{j=1}^{m} \frac{1}{\tau_i}} \\ n \geq m \end{array} \right. \Rightarrow \frac{M_{\text{LPT}}}{M_{\text{opt}}} \leq \frac{2m}{m+1} < 2$$

LPT: *Least Processing Time* scheduling heuristic.

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$\Rightarrow$ $p_{\text{fail}} \leq 2 \cdot \tilde{p}_{\text{fail}}$

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- Why?

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Remark: such a schedule is not always feasible (it just gives a lower bound).

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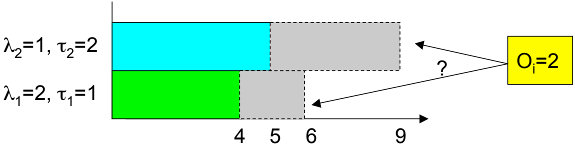

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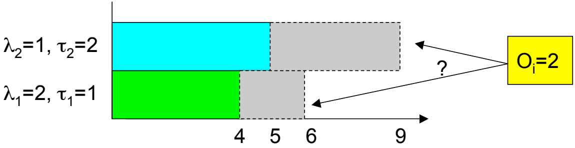

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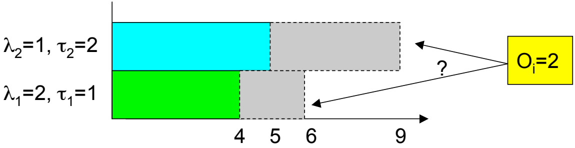

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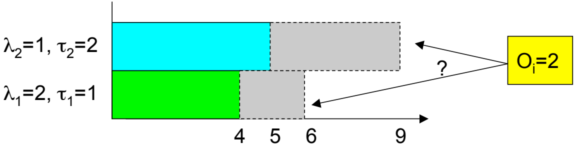

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Easy to extend to other heuristics (sufferage, etc.).

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Two ways top find a good trade-off:

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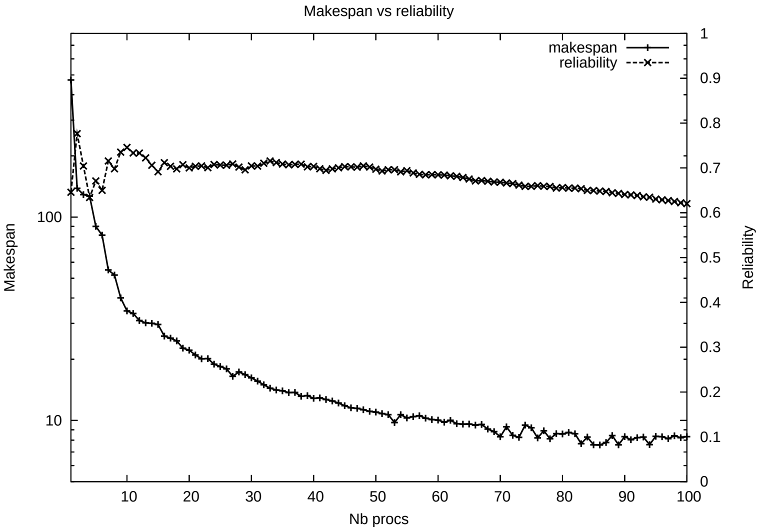
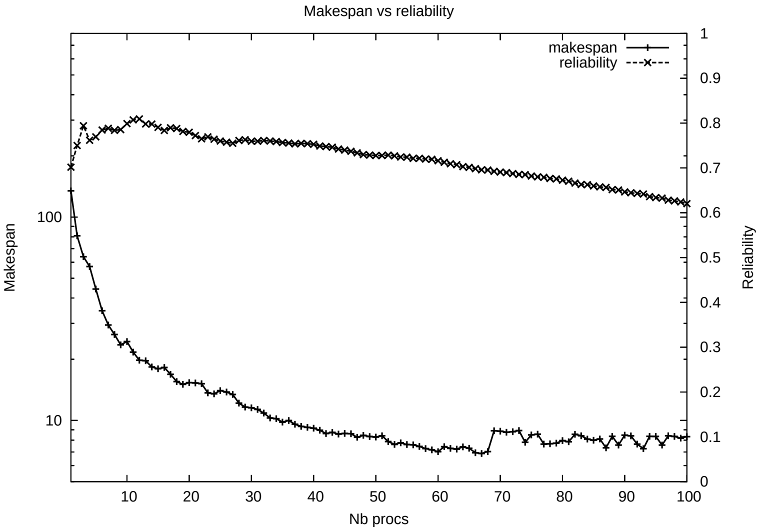
2. Use a trade-off variable α ($\alpha = 1$ switch to HEFT, $\alpha = 0$ switch to RHEFT).

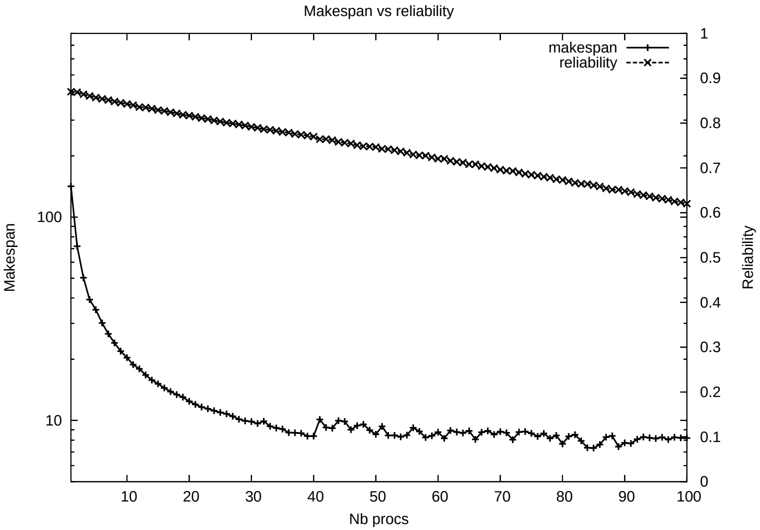
Ordering the processors: most reliable first



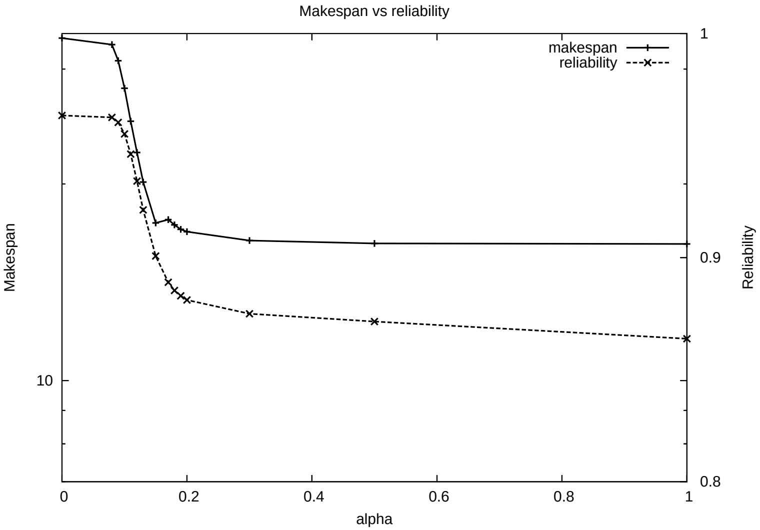
Ordering the processors: fastest first



Ordering the processors: smallest $\lambda\tau$ first



Trade-off variable



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- optimal algorithms for unitary independent tasks,

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- approximation algorithm for independent tasks ($n \geq m$),

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We have studied the problem of scheduling DAGs, with 2 objectives:

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Contribution:

- optimal algorithms for unitary independent tasks,
- approximation algorithm for independent tasks ($n \geq m$),
- simple way to generalize heuristics to this context,
- characterization of the role of the $\lambda\tau$ value.