A proof of st-comparison for polynomials of a stochastic matrix and how we can improve the accuracy of st-bounds *

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Abstract

We consider the pre-processing of stochastic matrices before computing a strong stochastic bound to improve the accuracy of Vincent's algorithm. Stochastic bounds are a promising technique for performance evaluation of networks as we can verify QoS statements on simpler models. We proved that applying some polynomials to the stochastic matrix of a discrete time Markov chain gives more accurate bounds. The proof is based on simple linear algebra techniques. We also present some numerical results for basic problems.

1 Introduction

Stochastic comparison of random variables have often been presented as a powerful technique in various areas of applied probability (see for instance the books by Kijima [7], Stoyan [13] and Shantikumar [12]). When models are too complex to be solved efficiently, the method allows to study simpler or smaller models with the guarantee that the results are upper or lower bounds. The distributions or rewards of the simpler model is smaller than the results of the original problem which remain unknown. Then we can directly prove that Quality of Service requirements are satisfied. Recently, several applications of this technique in practical problems of telecommunication engineering [10], [11] or in stochastic model checking [8] have been published.

Stochastic comparison of Markov Chains (MC in the following) is also possible and an algorithmic derivation of their stochastic bounds have been introduced by some of us [3, 4]. Sufficient conditions for the existence of stochastic comparison of two time-homogeneous MCs are given by the stochastic monotonicity and bounding properties of their one step transition probability matrices [6]. In [14], this idea is used to devise an algorithm that constructs a st-monotone upper bounding chain corresponding to a given continuous time Markov chain. Later, this algorithm was generalized to compute stochastic bounds which are simpler to solve based on some structural properties for Discrete-Time Markov Chains

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(DTMC for short in the following). Continuous time chains may be considered after uniformization. These algorithms have been surveyed recently [4] and a tool has been developed and presented [5].

The key problem is to obtain an accurate bound. We have observed that some pre-processing of the DTMC may lead to more accurate results. In [2], we have proved that we always improve the accuracy of the bound if we first apply a very simple linear transform. It is so simple that the amount of computations required is proportional to the number of non zero transitions in the matrix. Furthermore the transform can be computed on the fly while the matrix is generated from high level specifications (Stochastic Automata Network or Stochastic Petri Net for instance). Here we study more complex transforms which may be more expensive to compute but which may increases the accuracy of the bounding technique.

Even if the transforms we studied keep the steady-state distribution unchanged, we must say that we lost the transient comparison which is an important properties of the strong stochastic ordering. Thus these transforms must be used only when we study problems at steady-state and further transform have to be developed for analysis of transient distributions. Here, we assume that we accept to lose the transient bounds to improve the accuracy of steady state bounds.

The papers follows: in section 2, we briefly introduce Vincent's algorithm. Section 3 is devoted to the analysis of the transforms. In section 4, we give some numerical examples while in section 5, we present a formulation based on stochastic processes for some of our results.

2 Vincent's algorithm

First, we must recall the definition of st-ordering used in this paper. For further information on the stochastic comparison method, we refer the reader to [13]. The first definition is quite general, the second one only apply for discrete random variables and it leads to an algebraic statement of st-comparison.

Definition 1 *Let X and Y be random variables taking values on a totally ordered space. Then X is said to be less than Y in the strong stochastic sense, that is, $X \leqslant_{st} Y$ iff $E[f(X)] \leqslant E[f(Y)]$ for all nondecreasing functions f whenever the expectations exist.*

Definition 2 *Let X and Y be random variables taking values on the finite state space $\{1, 2, \dots, n\}$. Let p and q be probability distribution vectors such that*

$$p_j = Pr(X = j) \quad and \quad q_j = Pr(Y = j) \quad for \quad j = 1, 2, \dots, n.$$

Then X is said to be less than Y in the strong stochastic sense, that is, $X \leqslant_{st} Y$ iff

$$\sum_{j=k}^{n} p_j \leqslant \sum_{j=k}^{n} q_j \quad for \quad k = 1, 2, \dots, n.$$

As usual, $P_{i,*}$ refers to row i of P. For the sake of completeness, we recall the well-known fundamental theorem (see for instance Theorem 3.4 of [9, p. 355]) which states that monotonicity and comparability of the transition probability matrices of time-homogeneous MCs yield sufficient conditions for their stochastic comparison:

Theorem 1 *Let P and Q be stochastic matrices respectively characterizing time-homogeneous MCs $X(t)$ and $Y(t)$. Then $\{X(t),\ t \in \mathcal{T}\} \leqslant_{st} \{Y(t),\ t \in \mathcal{T}\}$ if*

- $X(0) \leqslant_{st} Y(0)$,

- *st-monotonicity of at least one of the matrices holds, that is,*

$$\textit{either} \quad P_{i,*} \leqslant_{st} P_{j,*} \quad \textit{or} \quad Q_{i,*} \leqslant_{st} Q_{j,*} \quad \forall i,j \quad \textit{such that} \;\; i \leqslant j,$$

- *st-comparability of the matrices holds, that is,* $P_{i,*} \leqslant_{st} Q_{i,*}\ \forall i$.

Unrolling the constraints, we can easily find the following algorithm to build a stochastic upper bound of a transition matrix.

Vincent's Algorithm to build an optimal st-monotone upper bounding DTMC Q corresponding to DTMC P of order n:

$q_{1,n} = p_{1,n}$;
for $i = 2, 3, \ldots, n$,
  $q_{i,n} = \max(q_{i-1,n}, p_{i,n})$;
for $l = n-1, n-2, \ldots, 1$,
  $q_{1,l} = p_{1,l}$;
  for $i = 2, 3, \ldots, n$,
    $q_{i,l} = \max(\sum_{j=l}^{n} q_{i-1,j}, \sum_{j=l}^{n} p_{i,l}) - \sum_{j=l+1}^{n} q_{i,j}$;

Some important properties have been proved about this algorithm. First, the bounding matrix may be reducible, even if the original matrix is irreducible. A new algorithm, IMSUB [3], has been introduced to fix this problem. But the original algorithm has an optimality property:

Property 1 *[14] Let U be another st-monotone upper bounding DTMC for P. Then Q is optimal in the sense that $Q \leqslant_{st} U$.*

However, this property does not mean that the bound on the distribution cannot be improved. We have studied the following idea: first pre-process the matrix with a transform which does not affect the steady-state distribution. Then, apply the bounding algorithm. For instance, in [2] we have considered a linear transform $t_\delta(P) = \delta Id + (1-\delta)P$. And we have proved that, for all δ:

$$\pi_P <_{st} \pi_{r^{-1}v(t_\delta(P))} <_{st} \pi_{r^{-1}v(P)}$$

where $r^{-1}v(R)$ is the matrix obtained when we apply Vincent's algorithm on matrix R. This notation will be explained in the next section. We have also shown that $\delta = 1/2$ provides the most accurate bound. Here, we will study more complex preprocessing (i.e. not linear) which gives even more accurate bounds. Thus, we will have another trade-off between accuracy and computational cost.

3 Stochastic Comparison : an algebraic approach

First, we define some sets and transform. In this chapter we use algebraic techniques but in the next section a stochastic interpretation of some of these results will be provided. Let $\mathcal{B}$ be the set of stochastic matrices with finite size (say n). Let P be an arbitrary ergodic stochastic matrix.

Definition 3 *Let $\mathcal{D}$ be the set of polynomials $\Phi()$ such that $\Phi(1) = 1$, Φ different of Identity, and all the coefficients of Φ are non negative.*

Lemma 1 *$\Phi(P)$ has the same steady-state distribution than P*

As the proof is quite simple, it is omitted here and will be added in the full paper.

Lemma 2 *Let P be an ergodic chain, and let $\Phi()$ be an arbitrary polynomial in $\mathcal{D}$, $\Phi(P)$ is ergodic.*

Proof : First, define formally $\Phi()$

$$\Phi(X) = \sum_{i=0}^{n} a_i X^i$$

The proof is based on these four arguments :

1. As $\Phi()$ is not the identity, there exist at least one degree $k > 0$ such that coefficient a_k is not zero.
2. $\Phi(1) = 1$ implies that $\sum_{i=0}^{n} a_i = 1$.
3. If P is ergodic, so is P^i for any positive value of i.
4. Finally, all $a_i \geqslant 0$ and $\Phi(P)$ is the convex sum of ergodic stochastic matrices. Thus, $\Phi(P)$ is ergodic.

Let us now introduce two operators on matrices which will provide a simple formulation for Vincent's algorithm.

Definition 4 *Let P be an arbitrary stochastic matrix, we define:*

- *r: the summation operator.*

$$r(P)[i,j] = \sum_{k=j}^{n} P[i,k]$$

- *Clearly, $r(P)$ is not a stochastic matrix. Thus, let $\mathcal{A}$ be the set of matrices defined by $r(\mathcal{B})$. Let Z be an arbitrary matrix in $\mathcal{A}$, $r^{-1}(Z)$ is now well defined.*

$$r^{-1}(Z)[i,n] = Z[i,n]$$

And

$$r^{-1}(Z)[i,j] = Z[i,j] - Z[i,j+1]$$

- *v : the following operator which transforms a stochastic matrix into a matrix in $\mathcal{A}$.*

$$v(P)[1,j] = \sum_{k=j}^{n} P[1,k] = r(P)[1,j]$$

And,

$$v(P)[i,j] = max_{m \leqslant i}\{\sum_{k=j}^{n} P[m,k]\} = max_{m \leqslant i}\{r(P)[m,j]\}$$

Property 2 *The strong stochastic comparison and the monotonicity of stochastic matrices can be defined through operator r:*

a- st-comparability: $P1 <_{st} P2 \quad \Leftrightarrow \quad r(P1) <_{el} r(P2) \ = \ r(P1)[i,j] \leqslant r(P2)[i,j] \quad \forall i,j$

b- $P1$ is st-monotone $\quad \Leftrightarrow \quad r(P1)[i,j] \leqslant r(P1)[i+1,j] \quad 1 \leqslant i \leqslant n-1$ *and* $\forall j$

A careful look at Vincent's algorithm shows that it is related to both operators we have defined.

Proposition 1 *The operator defined by Vincent's algorithm is $r^{-1}v$.*

Let us now turn to fundamental properties of the operators and some technical lemmas. Most of the properties we need here have already been proved in several papers. Thus we do not formerly prove them but we give some reference and key ideas. The fundamental lemma is the optimality property proved by Vincent and his co-author for continuous time Markov chains ([14]) or discrete time ones ([1]). We give here a new proof

Lemma 3 *Let $M = r^{-1}v(P)$ then M is the smallest st-monotone stochastic matrix larger than P.*
i.e. let Q be an arbitrary stochastic matrix such that

- *Q is st-monotone*
- *$P <_{st} Q$*

thus M is smaller than Q, i.e. $M <_{st} Q$

Proof : It follows from proposition 1 and the comparison of stochastic matrices (property 2), we must prove the following inequalities for each line i

$$v(P)[i,j] \leqslant r(Q)[i,j] \qquad \forall j \tag{1}$$

The proof is in induction in line number i.
For the first line $i = 1$, the operators r and v are equivalent. Since $P <_{st} Q$

$$v(P)[1,j] = r(P)[1,j] \leqslant r(Q)[1,j] \qquad \forall j$$

Let the inequalities 1 be satisfied for $i = k$ and we will prove them for $i = k+1$.
From induction hypothesis and the monotonicity of Q, we have

$$v(P)[k,j] = max_{m \leqslant k}\{r(P)[m,j]\} \leqslant r(Q)[k,j] \leqslant r(Q)[k+1,j] \qquad \forall j$$

Since $P <_{st} Q$

$$r(P)[k+1,j] \leqslant r(Q)[k+1,j] \qquad \forall j$$

Therefore

$$v(P)(k+1,j] = max_{m \leqslant k+1}\{r(P)[m,j]\} \leqslant r(Q)[k+1,j] \qquad \forall j$$

which completes the proof.
Before proceeding with polynomial of arbitrary degree, let us consider the simplest case: $\Phi(X) = X^2$.

Lemma 4 *if $P <_{st} Q$ and Q st-monotone then*

1. *Q^2 is st-monotone*

2. $P^2 <_{st} Q^2$

The proof is purely algebraic and is omitted. The first part follows directly from some properties stated in [6]. It is now possible to conclude for this simple example:

Theorem 2 *Consider $\Phi(X) = X^2$, (as $\Phi(1) = 1$, $\Phi(P)$ has the same steady-state distribution than P), for an arbitrary ergodic Markov chain P, we have :*

$$r^{-1}v\Phi(P) <_{st} \Phi(r^{-1}v(P))$$

Proof : applying lemma 4, with $Q = r^{-1}v(P)$, we get $\Phi(P) <_{st} \Phi(r^{-1}v(P))$. And this last matrix is st-monotone.

Now, we consider the application of operator $r^{-1}v$ to matrix $\Phi(P)$. According to Lemma 3 $r^{-1}v(\Phi(P))$ is the best st-monotone matrix greater than $\Phi(P)$. As $\Phi(r^{-1}v(P))$ is another st-monotone upper-bound, one must have

$$r^{-1}v\Phi(P) <_{st} \Phi(r^{-1}v(P))$$

And the proof is completed.

Instead of $\Phi(X) = X^2$ we now consider arbitrary polynomial in $\mathcal{D}$ and we first generalize Lemma 4.

Lemma 5 *Let Φ be an arbitrary polynomial in $\mathcal{D}$, let P et Q be two stochastic matrices in $\mathcal{A}$, if $P <_{st} Q$ and Q st-monotone then*

1. *$\Phi(Q)$ is st-monotone*

2. *$\Phi(P) <_{st} \Phi(Q)$*

Proof : The first part of the proof use the same facts that lemma 2.

1. If Q is st-monotone, so is Q^i for any positive value of i.
2. Finally, all $a_i \geqslant 0$, $\sum_{i=0}^{n} a_i = 1$, and $\Phi(Q)$ is the convex sum of st-monotone stochastic matrices. Thus, $\Phi(Q)$ is st-monotone (quite a similar statement has been established in [6]).
3. The second part is purely algebraic.

Once the lemma is established the key ideas of the theorem remain.

Theorem 3 *Let Φ be an arbitrary polynomial in $\mathcal{D}$, we have*

$$r^{-1}v\Phi(P) <_{st} \Phi(r^{-1}v(P))$$

The proof follows exactly the same scheme we have shown for theorem 2 but we must use lemma 5 instead of lemma 4. Of course, this stochastic comparison between matrices allows to compare the steady state distribution due to the properties of matrices $\Phi(r^{-1}v(P))$ and $r^{-1}v\Phi(P)$.

Corollary 1 *Consider an arbitrary polynomial $\Phi()$ in $\mathcal{D}$ for an arbitrary ergodic Markov chain P, we have :*

$$\pi_P <_{st} \pi_{r^{-1}v\Phi(P)} <_{st} \pi_{r^{-1}v(P)}$$

Proof : First remember that $\Phi(P)$ is a ergodic matrix (lemma 2). We apply theorem 3 to compare the matrices $r^{-1}v\Phi(P)$ and $\Phi(r^{-1}v(P))$ As $r^{-1}v\Phi(P)$ is monotone by construction, we can apply the fundamental theorem about Markov chain comparison. Thus

$$\pi_{r^{-1}v\Phi(P)} <_{st} \pi_{\Phi(r^{-1}v(P))}$$

But applying lemma 1, matrices $\Phi(r^{-1}v(P))$ and $r^{-1}v(P)$ have the same st-st distribution.

$$\pi_{r^{-1}v\Phi(P)} <_{st} \pi_{r^{-1}v(P)}$$

Similarly, we have :

$$\pi_{\Phi(P)} <_{st} \pi_{r^{-1}v\Phi(P)}$$

As $\pi_{\Phi(P)} = \pi_P$ the proof is completed.

We show in the next section that the accuracy of the bound is improved by this simple transform. And the improvements are better than the ones obtained by the linear transform we have already shown [2]. Clearly, these transforms imply more computations. At least, we must consider a polynomial with degree equals to 2. Therefore we must consider a sparse matrix multiplication or the generation of P^2 instead of P from a high level specification. This is not that difficult because the matrices we consider are usually extremely sparse. If the state number is n and the average number of transitions per state is d, the matrix multiplication has complexity in $O(n\ d^2)$. From a high level specification based on states and events, the generation of P has complexity $O(n\ d)$ while generation of P^2 requires only $O(n\ e^2)$ steps (e is the number of events).

In [2], we have proved that the accuracy due to the transform $\delta Id + (1-\delta)P$ increases with δ until the matrix becomes Row Diagonal Dominant (RDD). Thus the transform P/2+Id/2 provides the most accurate results because the matrix obtained is clearly RDD for all P. Here we do not have a similar optimality result and we are still investigating this question. However it is possible to obtain some simple results due to composition of polynomials.

Corollary 2 (Composition with P/2 +Id/2) *Let $\Phi(X)$ be an arbitrary polynomial in $\mathcal{D}$, the transform $\Psi(X) = \Phi(X)/2 + Id/2$ is also in $\mathcal{D}$, and the bound obtained after a pre-processing by Ψ is smaller or equal in distribution to the bound obtained after a pre-processing by Φ:*

$$\pi_P <_{st} \pi_{r^{-1}v\Psi(P)} <_{st} \pi_{r^{-1}v\Phi(P)}$$

Corollary 3 (General Composition) *Let $\Phi(X)$ and $\Psi(X)$ be arbitrary polynomials in $\mathcal{D}$, we have for arbitrary stochastic matrix P*

$$\pi_P <_{st} \pi_{r^{-1}v\Psi(\Phi(P))} <_{st} \pi_{r^{-1}v\Phi(P)}$$

Finally, we can generalize the theorem to arbitrary transform assuming that they satisfy the two basic properties we have considered in lemmas 5 and 1.

Theorem 4 *Let P be an arbitrary stochastic matrix, let ζ be a transform such that*

- *$\zeta(P)$ is stochastic for all P.*
- *if P is ergodic then $\zeta(P)$ is ergodic and $\pi_P = \pi_{\zeta(P)}$*
- *if P is st-monotone, so is $\zeta(P)$.*
- *if $P <_{st} Q$ then $\zeta(P) <_{st} \zeta(Q)$*

Then $r^{-1}v(\zeta(P)) <_{st} \zeta(r^{-1}v(P))$ and $\pi_P <_{st} \pi_{r^{-1}v(\zeta(P))} <_{st} \pi_{\zeta(r^{-1}v(P))}$.

4 Examples

First, we consider stochastic matrix $P = \begin{pmatrix} 0.1 & 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.4 & 0 \\ 0.2 & 0.1 & 0.3 & 0.4 \end{pmatrix}$ and we study the polynomials $\phi(X) = X/2+1/2$ and $\psi(X) = X^2/2+1/2$. Remember that polynomial $\phi()$ is proved to provided the best bound for the polynomials with degree 1. First, let us compute $\phi(P)$ and $\psi(P)$.

$$\phi(P) = \begin{pmatrix} 0.55 & 0.1 & 0.2 & 0.15 \\ 0.1 & 0.65 & 0.1 & 0.15 \\ 0.05 & 0.25 & 0.7 & 0 \\ 0.1 & 0.05 & 0.15 & 0.7 \end{pmatrix} \quad \psi(P) = \begin{pmatrix} 0.575 & 0.155 & 0.165 & 0.105 \\ 0.08 & 0.63 & 0.155 & 0.135 \\ 0.075 & 0.185 & 0.65 & 0.09 \\ 0.075 & 0.13 & 0.17 & 0.625 \end{pmatrix}$$

Then, we apply operators v and r^{-1} to obtain the bounds :

$$r^{-1}v\phi(P) = \begin{pmatrix} 0.55 & 0.1 & 0.2 & 0.15 \\ 0.1 & 0.55 & 0.2 & 0.15 \\ 0.05 & 0.25 & 0.55 & 0.15 \\ 0.05 & 0.1 & 0.15 & 0.7 \end{pmatrix} \quad r^{-1}v\psi(P) = \begin{pmatrix} 0.575 & 0.155 & 0.165 & 0.105 \\ 0.08 & 0.63 & 0.155 & 0.135 \\ 0.075 & 0.185 & 0.605 & 0.135 \\ 0.075 & 0.13 & 0.17 & 0.625 \end{pmatrix}$$

And,

$$r^{-1}v(P) = \begin{pmatrix} 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$$

Finally, we compute the st-st distribution for the bounds and the initial matrix :

$$\pi_P = (0.1530, 0.3025, 0, 3167, 0.2278)$$

$$\pi_{r^{-1}v(P)} = (0.1, 0.2, 0, 3667, 0.3333)$$

$$\pi_{r^{-1}v\phi(P)} = (0.1259, 0.2587, 0, 2821, 0.3333)$$

$$\pi_{r^{-1}v\psi(P)} = (0.1530, 0.2997, 0, 2916, 0.2557)$$

Clearly, bounds obtained by polynomial ψ are the more accurate. In the fill paper, we will consider a more applied model of a telecommunication system.

5 A stochastic approach

Let us know turn to a stochastic approach of the same result. Consider a homogeneous Markov chain $X = \{X_n\}_{n\in\mathbb{N}}$ which takes values in $\mathbb{N}$. This Markov chain is supposed to be positive recurrent, its stationary distribution is denoted by π_X.

Consider now an independent and identically distributed random sequence of integers $\{\tau_n\}_{n\in\mathbb{N}}$ stochastically independent of X. Define the subchain X^τ of X by

$$X_0^\tau = X_0;$$

$$X_n^\tau = X_{\tau_1+\cdots+\tau_n}.$$

Provided that $\mathbb{P}(\tau_i > 0) > 0$, the stochastic process X^τ, is a Markov chain. Its probability transition matrix is simply defined by $\mathbb{E}P^\tau$. It is clear that this chain is recurrent because

$$\lim_{n\to+\infty} \mathbb{P}(X_n^\tau = j | X_0^\tau = i) = \lim_{n\to+\infty} \mathbb{P}(X_n = j | X_0 = i) = \pi_j.$$

The term in the left limit is the limit of a subsequence of the converging sequence $\mathbb{P}(X_n = j|X_0 = i)$. This ensures the divergence of the potential matrix of X^τ. Following, it has the same stationary distribution $\pi_{X^\tau} = \pi_X$.

Recall that a Markov chain could also be constructed from a uniform random sequence on $[0,1[$, $\{U_n\}_{n\in\mathbb{N}}$. It is sufficient to define for each state i, the cumulative distribution function $F(i,j) = \mathbb{P}(X_1 \leqslant j|X_0 = i)$, then the generalized inverse of F at (i,u), $u \in [0,1[$ is given by

$$F^{-1}(i,u) = \max\{j : F(i,j) < u\}.$$

Then the stochastic process $\overline{X} = \{\overline{X_n}\}_{n\in\mathbb{N}}$ defined by the recurrence relation

$$\overline{X_n} = F^{-1}(\overline{X_{n-1}}, U_{n+1})$$

is a Markov chain. Moreover X and $\overline{X}$ have the same distribution.

The key point of Strassen's theorem is to rely the stochastic ordering $\leqslant_{st}$ with a coupling argument. Following this idea, we observe that a Markov chain is monotonous if and only if for all $u \in [0,1[$, $F^{-1}(i,u)$ is an increasing function of i. A Markov chain Y, with cdf G is a monotonous stochastic upper bound if and only if

$$F^{-1}(i,u) \leqslant G^{-1}(i,u) \leqslant G^{-1}(i+1,u) \leqslant G^{-1}(i+2,u) \leqslant \cdots, \text{for all } i \in \mathbb{N} \text{ and } u \in [0,1[.$$

Now associate to a Markov chain X the set $\mathcal{M}_X$ of all the monotonous upper bounds of X. It is clear that $(\mathcal{M}_X, \wedge)$ is lattice-structured with $\wedge$ the minimum operator on functions. More precisely if $(Y_1, Y_2) \in \mathcal{M}_X$ with respectively G_1 and G_2 as cdf. Define G by $G^{-1}(i,u) = min(G_1^{-1}(i,u), G_2^{-1}(i,u))$ and Y the associated coupled Markov chain. Then Y is monotonous, $Y \leqslant_{st} Y_1$, $Y \leqslant_{st} Y_2$ and $X \leqslant_{st} Y$, so $Y \in \mathcal{M}_X$. Because $\mathcal{M}_X$ is bounded beneath, there exist a unique minimum element in $\mathcal{M}_X$ denoted X_* (the corresponding coupled Markov chain). If we remark that $(X_*)^\tau$ is also monotonous because the composition of increasing function is an increasing function, we deduce that $(X_*)^\tau$ belongs to $\mathcal{M}_{X^\tau}$ and is greater than the minimal monotonous upper bound X_*^τ of X^τ.

Going throw the stationary regime we get

$$\pi_x \leqslant_{st} \pi_{X_*^\tau} \leqslant_{st} \pi_{(X_*)^\tau} = \pi_{X_*}.$$

6 Conclusion

The theory for Markov chains comparison is well-stated for a long time. The algorithmic aspects of this theory are still under development but it allows the automatic derivation of bounds in a software tools [5]. Improving the accuracy is one of the major problem as the quality of the bound may vary a lot due to several choices during the modeling process. We must now find some heuristics to help the users during the model design and the resolution process. The result we obtain here has a lot of algorithmic applications and it will certainly be the key issue for several heuristics in the future.

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